Monthly Contest 3
Due January 31, 2020

Instructions

This contest consists of 6 problems, each worth 5 points. Problems may not be of equal difficulty. Please write your solution to every problem on a separate sheet of paper. If you use more than one sheet for a specific problem, please staple the sheets together. Do NOT staple together solutions to different problems. It would be preferred if you used 8.5 by 11 printer paper to write your solutions on and leave a small margin around the edges. On the top of each sheet, include the contest number, problem number, and page number and the total number of pages of the problem's solution (not the total pages of all the solutions).

DO NOT put your name anywhere on your solutions. Doing so will result in instant disqualification. When you turn your solutions in, you will be asked to write a student code (which will be provided on the due date) on your solutions.

You must justify all answers to receive full credit. Answers without justification will receive few, if any, points. Partial credit will be given for significant progress into solving a problem. Leave answers in exact form unless otherwise specified. Do not feel discouraged if you cannot solve all the problems. It is already a great accomplishment if you are able to solve one of these problems. You are NOT allowed to consult or talk to anyone else about the problems, whether in person, on the phone, via e-mail, on forums, or any other means of communication. **You may not view any website (including forums) unless otherwise stated in the problem.**

Problems

1. You have a calculator with 3 operations:

   (a) You can determine if two numbers are equal.

   (b) You can add two numbers together.

   (c) You can find the roots of $x^2 + ax + b$, and if they exist.

   However, this calculator is quite strange. It cannot take direct input and can only used values it has stored in "memory". All numeric results of these operations are stored in memory. (For example, if you had a 4 in memory, you could use (b) on 4 and 4 to get 8 and your memory would contain 4 and 8). If you start with a number k in memory, how do you determine if k is 1?

2. Prove that there does not exist integer n such that $n^7 + 7$ is a square.

3. Each vertex of a regular 1997-gon is labeled with an integer, such that the sum of the integers is 1. Starting at some vertex, we write down the labels of the vertices reading counterclockwise around the polygon. Is it always possible to choose the starting vertex such that the sum of the first k integers written down is positive for $k = 1, ..., 1997$?

4. A checker is placed in each of the unit squares of a $n \times n$ square, which is part of an infinitely large board in all directions. A move is the process of selecting two checkers located in unit squares sharing a side, and using one of the checkers to jump over the other checker into an empty square adjacent by a side to the square in which the other checker is located. The checker that has been jumped over is removed from the board. After several moves it will be impossible to make a move. Prove that this will happen after at least $\lfloor \frac{n^2}{3} \rfloor$ moves for all even n. (Challenge: prove for all n)

5. In SJMC, some students are friends. Friendship is always mutual. Call a group of students a clique if all pairs of them are friends. (Any group of fewer than two is also clique.) The number of members of a clique is called its size. Given that, in this competition, the largest size of a clique is even, prove that the students can be arranged into two rooms such that the largest size of a clique in one room is the same as the largest size of a clique contained in the other room.

6. Let $f\colon \mathbb{N} \to \mathbb{N}$ be a function satisfying

   (a) For every $n \in \mathbb{N}$, $f(n + f(n)) = f(n)$.

   (b) For some $n_0 \in \mathbb{N}$, $f(n_0) = 1$.

   Show that $f(n) = 1$ for all $n \in \mathbb{N}$.