Monthly Contest 1
Due Noverber 8, 2019

Instructions

This contest consists of 6 problems, each worth 5 points. Problems may not be of equal difficulty. Please write your solution to every problem on a separate sheet of paper. If you use more than one sheet for a specific problem, please staple the sheets together. Do NOT staple together solutions to different problems. It would be preferred if you used 8.5 by 11 printer paper to write your solutions on and leave a small margin around the edges. On the top of each sheet, include the contest number, problem number, and page number and the total number of pages of the problem's solution (not the total pages of all the solutions).

DO NOT put your name anywhere on your solutions. Doing so will result in instant disqualification. When you turn your solutions in, you will be asked to write a student code (which will be provided on the due date) on your solutions.

You must justify all answers to receive full credit. Answers without justification will receive few, if any, points. Partial credit will be given for significant progress into solving a problem. Leave answers in exact form unless otherwise specified. Do not feel discouraged if you cannot solve all the problems. It is already a great accomplishment if you are able to solve one of these problems. You are NOT allowed to consult or talk to anyone else about the problems, whether in person, on the phone, via e-mail, on forums, or any other means of communication. **You may not view any website (including forums) unless otherwise stated in the problem.**

Problems

1. The itsy bitsy spider goes up the water spout. Every second, the spider climbs up 1 cm. However, after every 10 moves the spider makes, there is a $\frac{1}{2}$ chance that it will rain and the spider will be washed back to the start. Given that the chance of the spider getting to the top in 1 minute is at least half, find the maximum possible length of the water spout.

2. Find all ordered pairs of integers (x, y) such that

$$x + y = x^2 - xy + y^2$$

3. Circle ω is goes through the incenter I of $\triangle ABC$ and is tangent to $\overline{AB}$ at A. ω intersects line BC at D and E, such that D is on line segment $\overline{BC}$. Prove that the intersection of IC and ω is equidistant from D and E.

4. For all real b, let $f(b)$ be the maximum value of

$$|sin(x) + \frac{2}{3 + sin(x)}|$$

over all real x. Find the minimum value of $f(b)$ over all real b.

5. How many pairs (x, y) of positive integers with $x \leq y$ satisfy $gcd(x, y) = 5!$ and $lcm(x, y) = 50!$?

6. Let A be the set of positive integers that can be represented by $a^2 + 2b^2$ with integers a, b, and $b \neq 0$. Prove that if $p^2 \in A$, then $p \in A$, if p is prime.