San Jose Math Circle: Graph Coloring

October 2018

(1) A *coloring* of a map is a coloring of the different regions (or countries, however you view your map) in such a way that regions that share parts of their boundary get different colors. (If two countries only share a few points on their borders, such as Utah and New Mexico on a map of the US, they can get the same color.) The *chromatic number* of a map is the minimum number of colors we need to color it. Construct (many) maps with chromatic number 1, 2, 3, 4, and 5.

(2) A *graph* is a mathematical concept consisting of *nodes* (which you can just think of as points) and *edges* between these nodes (which you can think of as curves connecting the node points). If there is an edge between two nodes, we say that these nodes *share* the edge, or that the two nodes are *adjacent*. Every map gives rise to a graph (called the *dual graph* of the map) in the following sense: each region in the map gives rise to a node in the graph, and two nodes share an edge if the corresponding regions share part of their boundaries. A *coloring* of a graph is a coloring of its nodes such that any two adjacent nodes get distinct colors. The *chromatic number* of a graph is the smallest number of colors needed to color the graph.

   (a) Convince yourself that a map coloring corresponds exactly to a coloring of the corresponding graph.
   (b) Go through your examples in (1) again in terms of the graphs that come with your maps.
   (c) What do you notice about the graphs that arise from maps?

(3) Here are four classes of graphs:

   - The *null graph* N_n consists of n nodes and no edges.
   - The *complete graph* K_n consists of n nodes with all possible edges between them.
   - The *path graph* L_n consists of n nodes with edges that form a line segment.
   - The *cycle* C_n consists of n nodes with edges that form a circle.

   Compute the chromatic numbers of these graphs.

(4) Given a graph G and a positive integer k, let $c_G(k)$ be the number of colorings of G that use at most k colors. Compute $c_G(k)$ for the four classes mentioned above. What do you notice about

   (a) the leading coefficients?
   (b) the second leading coefficients?
   (c) the constant term?
   (d) the highest degree?

(5) Fix an edge e in a graph G. Denote by $G \setminus e$ be the graph you get from G by *deleting* e, and by $G \cdot e$ the graph you get from G by *contracting* e (i.e., identifying the two nodes that share e). Find a relationship of $c_G(k)$, $c_{G \setminus e}(k)$, and $c_{G \cdot e}(k)$.

(6) What graphs do you obtain by continuously deleting and contracting edges of a given graph G? Use this observation and the identity you found in (5) to prove that $c_G(k)$ is a polynomial in k.

(7) Use a similar reasoning to prove your observations from (3).

(8) Experiment with the numbers $c_G(-1)$ for different examples of graphs G. Can you guess what they count? (Hint: look at *acyclic orientations* of G, i.e., give each edge a direction, in such a way that you can't see any coherently oriented cycle.) Try to prove your assertion.