Monthly Contest #3: Upper Division

Due: March 8, 2019

San Jose Math Circle

January 2019

Instructions: This contest consists of 5 problems, each worth 7 points. Problems may not be of equal difficulty. Please write your solution to every problem on a separate sheet of paper. If you use more than one sheet for a specific problem, please staple the sheets together. Do NOT staple together solutions to different problems. It would be preferred if you used 8.5 by 11 printer paper to write your solutions on and leave a small margin around the edges. On the top of each sheet, include the contest number, problem number, and page number and the total number of pages of the problems solution (not the total pages of all the solutions). DO NOT put your name anywhere on your solutions. Doing so will result in instant disqualification. When you turn your solutions in, you will be asked to write a student code (which will be provided on the due date) on your solutions.

You must justify all answers to receive full credit. Answers without justification will receive few, if any, points. Partial credit will be given for significant progress into solving a problem. Leave answers in exact form unless otherwise specified. Do not feel discouraged if you cannot solve all the problems. It is already a great accomplishment if you are able to solve one of these problems. You are NOT allowed to consult or talk to anyone else about the problems, whether in person, on the phone, via e-mail, on forums, or any other means of communication. You may not view any book or website (including forums) unless otherwise stated in the problem.

Problems

Problem 1. The natural number n can be replaced by ab if $a+b=n$, where a and b are natural numbers. Can the number 2001 be obtained from 22 after a sequence of such replacements?

Problem 2. Values $a_1, \ldots, a_{2013}$ are chosen independently and at random from the set $\{1, \ldots, 2013\}$. What is the expected number of distinct values in the set $\{a_1, \ldots, a_{2013}\}$?

Problem 3. Let $n > 0$ be an integer. We are given a balance and n weights of weight $2^0, 2^1, \cdots, 2^{n-1}$. We are to place each of the n weights on the balance, one after another, in such a way that the right pan is never heavier than the left pan. At each step we choose one of the weights that has not yet been placed on the balance, and place it on either the left pan or the right pan, until all of the weights have been placed. Determine the number of ways in which this can be done.

Problem 4. Tanya chose a natural number $X \leq 100$, and Sasha is trying to guess this number. He can select two natural numbers M and N less than 100 and ask about $\gcd(X+M, N)$. Show that Sasha can determine Tanya's number with at most seven questions.

Problem 5. Find all positive integers n such that n^2 divides 2^n+1.