Monthly Contest #1: Upper Division

Due: December 7, 2018

San Jose Math Circle

November 2018

Instructions: This contest consists of 5 problems, each worth 7 points. Problems may not be of equal difficulty. Please write your solution to every problem on a separate sheet of paper. If you use more than one sheet for a specific problem, please staple the sheets together. Do NOT staple together solutions to different problems. It would be preferred if you used 8.5 by 11 printer paper to write your solutions on and leave a small margin around the edges. On the top of each sheet, include the contest number, problem number, and page number and the total number of pages of the problems solution (not the total pages of all the solutions). DO NOT put your name anywhere on your solutions. Doing so will result in instant disqualification. When you turn your solutions in, you will be asked to write a student code (which will be provided on the due date) on your solutions.

You must justify all answers to receive full credit. Answers without justification will receive few, if any, points. Partial credit will be given for significant progress into solving a problem. Leave answers in exact form unless otherwise specified. Do not feel discouraged if you cannot solve all the problems. It is already a great accomplishment if you are able to solve one of these problems. You are NOT allowed to consult or talk to anyone else about the problems, whether in person, on the phone, via e-mail, on forums, or any other means of communication. You may not view any book or website (including forums) unless otherwise stated in the problem.

Problems

Problem 1. Find the remainder when $2018^{2017^{2016}}$ is divided by 1000.

Problem 2. Let $f(n) = \varphi(n^3)^{-1}$, where $\varphi(n)$ denotes the number of positive integers not greater than n that are relatively prime to n. Find

$$\frac{f(1) + f(3) + f(5) + \dots}{f(2) + f(4) + f(6) + \dots}.$$

The following fact may be useful: if a and b are coprime integers, then $\varphi(ab) = \varphi(a) \cdot \varphi(b)$.

Problem 3. Find all functions $f : \mathbb{N} \to \mathbb{N}$ (in other words, functions that take positive integers to positive integers) satisfying

$$f(m + n) = f(m) + f(n) + mn$$

for all positive integers m, n.

Problem 4. A trapezoid $ABCD$ with bases AD and BC is such that $AB = BD$. Let M be the midpoint of DC. Prove that $\angle MBC = \angle BCA$.

Problem 5. Let n be an positive integer. Find the smallest integer k with the following property; Given any real numbers $a_1, \cdots, a_d$ such that $a_1 + a_2 + \cdots + a_d = n$ and $0 \leq a_i \leq 1$ for $i = 1, 2, \cdots, d$, it is possible to partition these numbers into k groups (some of which may be empty) such that the sum of the numbers in each group is at most 1.