2016-17 Monthly Contest 6 Solutions

Note: The following solutions are not the only solutions possible. We encourage you to seek other solutions, and perhaps yours will be more elegant than ours!

1. Let $\mathbb{N}$ be the set of positive integers. Find all strictly increasing functions f : $\mathbb{N} \rightarrow \mathbb{N}$ such that $f(2) = 2$ and $f(m)f(n) = f(mn)$ for all positive integers m and n.

   Solution:

   We can first show that $f(1) = 1$. Given that $f(m)f(n) = f(mn)$, we can let $m = 1$ and $n = 2$. As a result we get

   $$f(1)f(2) = f(2)$$
   $$f(1) \cdot 2 = 2$$
   $$f(1) = 1.$$

   Now let's try to find other values that satisfy f. First, if we plug $m = 2$ into the condition, then we get $f(2n) = 2f(n)$. This inspires us to hypothesize that $f(2^k) = 2^k$ for all positive integers k.

   Lemma. *$f(2^k) = 2^k$ for all positive integers k.*

   We will prove this claim using induction. For our base case, set $k = 0$. $f(2^0) = f(1) = 1 = 2^0$, so the lemma holds true for the base case.

   Now we can suppose that $f(2^{k-1}) = 2^{k-1}$. We want to show that $f(2^k) = 2^k$. Using the given condition, we get

   $$f(2^{k-1}) \cdot f(2) = f(2^{k-1} \cdot 2)$$
   $$2^{k-1} \cdot 2 = f(2^k)$$
   $$2^k = f(2^k). \square$$

   We haven't used the strictly increasing condition yet; it is extremely important here, because it allows us to use a bounding argument. Suppose we want to show that $f(n) = n$ for a particular n that is not a power of 2, and take the k with $2^k < n < 2^{k+1}$. Then we have

   $$2^k = f(2^k) < f(2^k + 1) < ... < f(n) < ... < f(2^{k+1}) = 2^{k+1}.$$

   We have to fit $2^{k+1} - 2^k + 1$ values of f strictly between 2^k and 2^{k+1}. There are only $2^{k+1} - 2^k + 1$ available numbers to use, so we must use each of them in order. Therefore, $f(n) = n$, as desired.

   In conclusion, the only possible function is $f(n) = n.\square$

2. 3 students each from Hilbert Middle School, Galois Middle School, and Noether Middle School sit in a row of 9 seats. In how many ways can we seat the students if no three students from the same school are seated next to each other?

   Solution:

   Let the letters H, G, and N denote fan from Hilbert Middle, Galois Middle, and Noether Middle, respectively. Then the seating of the students can be represented by the arrangement of the nine letters H, H, H, G, G, G, N, N, and N. Let S_H be the set of sequences of letters that result in three H's appearing next to each other; S_G and S_N are defined similarly. Define the function $\eta(S)$ as the number of arrangements in a set S.

   We will solve this question using complementary counting. By PIE (Principle of Inclusion-Exclusion), the number of sequences where any of the three letters H, G, or N appear next to each other is

$$\begin{aligned}\eta(S_H \cup S_G \cup S_N) = {} & \eta(S_H) + \eta(S_G) + \eta(S_N) \\ & - \eta(S_H \cap S_G) - \eta(S_G \cap S_N) - \eta(S_H \cap S_N) \\ & + \eta(S_H \cap S_G \cap S_N).\end{aligned}$$

   We then count the individual terms. $\eta(S_H)$ is the number of arrangements where the H's are next to each other. We can consider the three H's as one single block. Then, the number of arrangements of the single block HHH, 3 G's, and 3 N's is

$$\frac{7!}{1!3!3!} = 140.$$

   Once the positions of the letters have been determined, the students are distinct so the number of ways to seat the students from each school is 3!. Thus $\eta(S_H) = \eta(S_G) = \eta(S_N) = 140 \cdot 3! \cdot 3! \cdot 3! = 30240$.

   Next, we count $\eta(S_H \cap S_G)$. We consider both HHH and GGG as two singular blocks. Thus the number of arrangmenets with a HHH block, a GGG block, and three N's is

$$\frac{5!}{1!1!3!} = 20.$$

   There are 3! ways to seat the students from each school, so $\eta(S_H \cap S_G) = \eta(S_G \cap S_N) = \eta(S_H \cap S_N) = 20 \cdot (3!)^3 = 4320$.

Lastly, we need to count $\eta(S_H \cap S_G \cap S_N)$, which is just $(3!)^4 = 1296$. Therefore,

$$\eta(S_H \cup S_G \cup S_N) = 3 \cdot 30240 - 3 \cdot 4320 + 1296 = 79056.$$

Since we are counting the complement of what we want to find, we subtract 79056 from 9!=362880 total ways to get 362880-79056= $\boxed{283824}$ ways of seating the nine students so that no three students of the same school sit next to each other.

3. Find the minimum term in the sequence

$$\sqrt{\frac{7}{6}} + \sqrt{\frac{96}{7}}, \sqrt{\frac{8}{6}} + \sqrt{\frac{96}{8}}, ..., \sqrt{\frac{95}{6}} + \sqrt{\frac{96}{95}}.$$

Solution:

Every term in the sequence is of the form

$$\sqrt{\frac{n}{6}} + \sqrt{\frac{96}{n}}$$

for $7 \leq n \leq 95$.

Applying the AM-GM inequality, we get

$$\frac{\sqrt{\frac{n}{6}} + \sqrt{\frac{96}{n}}}{2} \geq \sqrt{\sqrt{\frac{n}{6}} \cdot \sqrt{\frac{96}{n}}} = \sqrt{\sqrt{\frac{96}{6}}} = 2.$$

Thus $\sqrt{\frac{n}{6}} + \sqrt{\frac{96}{n}} \geq 4$. By the rules of AM-GM, the equality holds when $\sqrt{\frac{n}{6}} = \sqrt{\frac{96}{n}}$, so $n = 24$.

Therefore, the minimum term in the sequence is $\boxed{\sqrt{\frac{96}{24}} + \sqrt{\frac{24}{6}}}$.

4. Prove that the sum of the squares of 3,4,5, or 6 consecutive integers is not a perfect square.

Solution:

For 3 consecutive integers:
Note that 3 consecutive integers will always be congruent to 0,1, and 2 (mod 3), in any order. When these numbers are squared, it results in 0,1, and 1 (mod 3) respectively, which add up to a sum of 2(mod 3).

Because it is impossible for a square to equal 2(mod3), as shown above, it is not possible for the sum of the squares of 3 consecutive integers to add up to a perfect square.

For 4 consecutive integers:
This is basically the same as for 3 consecutive integers. For 4 consecutive integers, they will always be congruent to 0,1,2, and 3 (mod 4), in any order. When these numbers are squared, it results in 0,1,0, and 1 (mod 4) respectively, which add up to a sum of 2(mod 4). Because it is impossible for a square to equal 2(mod4), as shown above, it is not possible for the sum of the squares of 4 consecutive integers to add up to a perfect square.

For 5 consecutive integers:
5 consecutive integers will always be congruent to 0,1,2,3, and 4 (mod 5), in any order. When these numbers are squared, it results in 0,1,4,4, and 1 (mod 5) respectively.
Let n be the middle of the 5 integers. When we square the integers, we get that
$(n-2)^2+(n-1)^2+n^2+(n+1)^2+(n+2)^2 = 5n^2+10$. This simplifies to x^2+2 (mod 5). Since x^2+2 (mod 5) must equal 0, $x^2 = 3$, which is not possible, as shown above.

For 6 consecutive integers:
Let the 6 consecutive integers be $(n-2)+(n-1)+n+(n+1)+(n+2)+(n+3)$. We then get that
$(n-2)^2+(n-1)^2+n^2+(n+1)^2+(n+2)^2 = 6n^2+6n+19$
This is congruent to $2n^2+2n+3$ (mod 4). Because $n(n+1)$ must be even, $2n^2+2n$ must be divisible by 4, leading to $2n^2+2n+3 \equiv 3$(mod 4). Because no square can equal 3(mod 4) (as shown previously), it is not possible for the sum of the squares of 6 consecutive integers to add up to a perfect square.

5. Find all the integer solutions of the system of equations:

$$ab + cd = -1 \quad (1)$$
$$ac + bd = -1 \quad (2)$$
$$ad + bc = -1 \quad (3)$$

Solution:

We begin by subtracting Equation 2 from Equation 1 and factoring the left side:

$$\begin{aligned} ab + cd - ac - bd &= 0 \\ a(b-c) - d(b-c) &= 0 \\ (a-d)(b-c) &= 0. \end{aligned}$$

That means $a = d$ or $b = c$. Without loss of generality, let's assume that $a = d$.

Substituting $a = d$ into Equations 2 and 3, we get

$$ab + ac = -1 \tag{4}$$
$$a^2 + bc = -1 \tag{5}$$

Subtracting the two equations above yields

$$a^2 + bc - ab - ac = 0.$$

This can be factored as $(a-c)(a-b) = 0$, so we can conclude that $a = c$ or $a = b$. Similarly, WLOG we can choose $a = c$.

Now, we substitute $a = c$ into Equation 4 to get $a^2 + ab = -1$. Thus

$$a^2 + ba + 1 = 0.$$

For a to be an integer, the discriminant $b^2 - 4$ must be a perfect square. That means b must be 2 or -2.

The complete corresponding solutions (a, b, c, d) are:

$$\begin{aligned} &(1, 1, 1, -2) \\ &(1, 1, -2, 1) \\ &(1, -2, 1, 1) \\ &(-2, 1, 1, 1) \\ &(2, -1, -1, -1) \\ &(-1, 2, -1, -1) \\ &(-1, -1, 2, -1) \\ &(-1, -1, -1, 2). \end{aligned}$$