2016-17 Monthly Contest 5 Solutions

Note: The following solutions are not the only solutions possible. We encourage you to seek other solutions, and perhaps yours will be more elegant than ours!

1. Let n be a positive integer. Prove that $5n^2 + 4$ or $5n^2 - 4$ is a perfect square if and only if n is a Fibonacci number.

   Solution:

   Let m and n be positive integers such that $5n^2 \pm 4 = m^2$. Taking this equation mod 2, we get $n^2 \equiv m^2 \pmod 2$, which means m and n have the same parity. Let $x = \frac{m-n}{2}$ and $y = n$. Then, $m = 2x + y$. Substituting these values, we get

   $$5y^2 \pm 4 = (2x + y)^2,$$

   which simplifies to

   $$y^2 - xy - x^2 = \pm 1.$$

   Lemma. *If x and y are solutions to $y^2 - xy - x^2 = \pm 1$, then y is a Fibonacci number. Conversely, if y is a Fibonacci number, we can find some Fibonacci number x such that $y^2 - xy - x^2 = \pm 1$.*

   Proof of lemma: if we set $r = y$ and $s = x - y$, then

   $$\begin{aligned} r^2 - sr - s^2 &= y^2 - (x-y)y - (x-y)^2 \\ &= y^2 - xy + y^2 - x^2 + 2xy - y^2 \\ &= -x^2 + xy + y^2 \\ &= -(x^2 - xy - y^2), \end{aligned}$$

   so

   $$(r^2 - sr - s^2)^2 = (x^2 - xy - y^2)^2 = 1.$$

   Thus, $(r, s) = (y, x - y)$ is also a solution to $(r^2 - sr - s^2)^2 = 1$.

   Thus, when given a solution (x, y) in positive integers, we can define a sequence as follows: let $n_1 = x, n_2 = y$, and for all positive integers $k \geq 3$, if $n_{k-1} > 1$, then

   $$n_k = n_{k-1} - n_{k-2}.$$

   This sequence of positive integers is clearly decreasing and terminates when $n_k = 1$.

From our observation above, $(r, s) = (n_{k-1}, n_k)$ is a solution to $(r^2 - sr - s^2)^2 = 1$ for all $k \geq 2$. If $s = 1$, then the only solutions to $(r^2 - sr - s^2)^2 = 1$ in positive integers r are $r = 1$ and $r = 2$. Hence, if k is the index such that $n_k = 1$, then $n_{k-1} = 2$. By a straightforward induction argument, $n_1 = F_{k+1}$ and $n_2 = F_k$, where F_n denotes the n^{th} Fibonacci number. Thus, y is a Fibonacci number.

Conversely, if y is a Fibonacci number, we can use the above relation in reverse to find the desired Fibonacci number x. □

By the lemma above, we conclude that n is a Fibonacci number. Conversely, if $y = n$ is a Fibonacci number, then by the lemma we have a Fibonacci number x for which

$$y^2 - xy - x^2 = \pm 1.$$

Setting $m = 2x + y$, we get $5n^2 \pm 4 = m^2$.

2. Let P be an interior point of △ABC, and extend lines from the vertices through P to the opposite sides. Let AP = a, BP = b, CP = c, and let the extensions from P to the opposite sides all have length d. If $a + b + c = 40$ and $d = 4$, find abc.

   Solution:

   We can use a theorem where in △ABC, if cevians AD, BE, and CF meet at P, then

$$\frac{PD}{AD} + \frac{PE}{BE} + \frac{PF}{CF} = 1$$

   In order to prove this, let BD:DA = p:q and BD:DC = r:s. Form mass points psA, qsB, and qrC. Then balancing along the sides at the feet of the cevians gives mass points (qs+qr)D, (ps+qr)E, and (ps+qr)F. Reading the ratios from the triangle leaves us with:

$$\frac{PD}{AD} + \frac{PE}{BE} + \frac{PF}{CF} = \frac{ps}{ps + qs + qr} + \frac{qs}{ps + qs + qr} + \frac{qr}{ps + qs + qr} = 1$$

   If we substitute $d = 4$ into the equation, we get that $\frac{d}{a+d} + \frac{d}{b+d} + \frac{d}{c+d} = \frac{4}{a+4} + \frac{4}{b+4} + \frac{4}{c+4} = 1$. We can then multiply by the common denominator to get that

$$4[(a+4)(b+4)) + ((a+4)(c+4))((b+4)(c+4)] = (a+4)(b+4)(c+4)$$

If we expand and simplify this, we get that $abc = 16(a + b + c) + 128$. After substituting in $a + b + c$ we come to the answer that $abc = 768$.

3. Two spies, Alex and Bethany, must pass each other their secret IDs,which are integers in the range 1-1700. They meet at a river, where there is a pile of 26 indistinguishable stones. Starting with Alex, they take turns throwing a group of stones into the river. Each spy must throw at least one stone on his/her turn, until all the stones are gone.They observe all throws and leave when there are no more stones. No information is exchanged except the number of stones thrown at each turn. How can they exchange the numbers successfully?

   Solution:

   We will have Alex and Bethany each have 13 stones. Starting with Alex, each person will take a turn to throw at least 1 stone, until there are no stones left. Suppose each person only gets to throw k times.

   Let's first look at $k = 2$, so the sequence is first Alex throws at least one stone, then Bethany throws at least one stone, then Alex throws his remaining stones, and finally Bethany throws her remaining stones. How many ways can Alex throw a total of 13 stones in two turns, where he needs to throw at least 1 stone each turn? The answer is 12: the choices are (1, 12), (2, 11), ..., and (12,1). Thus in two turns, Alex and Bethany can each communicate an integer from the interval [1, 12] with each throwing sequence representing a distinct integer.

   For general k, we seek the number of positive integer solutions to $a_1 + a_2 + ... + a_k = 13$ for each person, where a_i represents the number of stones thrown on the i-th turn. The count is $\binom{12}{k-1}$ by stars and bars. To maximize this term, we pick $k = 7$¡ which gives us $\binom{12}{6} = 924$. So, if Alex and Bethany each throw stones for 7 turns, and finishes throwing all stones by the 7th turn, they can each communicate an integer from the interval [1, 924].

   Next, we loosen the constraint that each person must use up all 13 stones by the 7th turn; in other words, after the 7th turn, Alex or Bethany may still have stones left. There are several possible cases to choose how many total stones to throw individually: use only 7 stones, 8 stones, 9 stones, ..., 12 stones, and 13 stones. Using the stars and bars method once again, the total count becomes

$$\binom{6}{6} + \binom{7}{6} + \binom{8}{6} + ... + \binom{12}{6},$$

which can be simplified to $\binom{13}{7} = 1716$ by the Hockey Stick Identity. 1716 is larger than 1700, so it is possible for Alex and Bethany to exchange their IDs this way.

4. (AIME 1992, #6) For how many pairs of consecutive integers in the set

$$\{1000, 1001, 1002, ..., 2000\}$$

is no carrying required when the two integers are added?

Solution (adapted from AoPS):

Carrying means that the sum of two digits is greater than or equal to ten, so you need to "carry" the tens digit over. In this problem context, if carrying is needed to add two numbers $\overline{abcd}$ and $\overline{efgh}$, then $h + d \geq 10$, $c + g \geq 10$, or $b + f \geq 10$.

Consider $c \in \{0, 1, 2, 3, 4\}$. $\overline{1abc} + \overline{1ab(c+1)}$ has no carrying if $a, b \in \{0, 1, 2, 3, 4\}$. This gives $5^3 = 125$ possible options.

With $c \in \{5, 6, 7, 8\}$, there must be carrying. Consider $c = 9$. Selecting $a, b \in \{0, 1, 2, 3, 4\}$ will result in no carrying. This gives $5^2 = 25$ possible options. If $b = 9$, $a \in \{0, 1, 2, 3, 4, 9\}$ will result in no carrying. Thus, the final answer is $125 + 25 + 6 = \boxed{156}$.

5. Prove that for any integer $n > 1$

$$n! < (\frac{n+1}{2})^n.$$

Solution: By the AM-GM inequality, we get

$$\frac{1 + 2 + 3 + ... + (n-1) + n}{n} \geq \sqrt[n]{n!}. \tag{1}$$

We can eliminate the possibility of the left-hand side being equal to the right-hand side of inequality (1) because clearly the n terms are not equal ($1 \neq 2 \neq ... \neq n$). Thus, we can simplify inequality (1) to

$$\begin{aligned} \frac{1 + 2 + 3 + ... + (n-1) + n}{n} &> \sqrt[n]{n!} \\ \frac{n(n+1)}{2n} &> \sqrt[n]{n!} \\ \frac{n+1}{2} &> \sqrt[n]{n!} \\ (\frac{n+1}{2})^n &> \sqrt{n!}. \end{aligned}$$

We have proven that $n! < (\frac{n+1}{2})^n$, as desired. $\square$