

2016-17 Monthly Contest 4 Solutions

Note: The following solutions are not the only solutions possible. We encourage you to seek other solutions, and perhaps yours will be more elegant than ours!

- Let n be an odd integer greater than 1. Prove that the sequence $\binom{n}{1}, \binom{n}{3}, \binom{n}{5}, \dots, \binom{n}{n-2}$ contains an odd number of odd numbers.

Solution:

Note that the sequence is exactly half of the $(n + 1)^{th}$ row of the Pythagorean triangle without the 1's on the side. Since we know that the sum of each row of the Pythagorean triangle is $2^{\text{row number} - 1}$. This gives us the sum $\frac{1}{2}(2^n - 2) = 2^{n-1} - 1$, which is an odd number. Because the sum is an odd number, the sequence must contain an odd number of odd numbers. \square

- Abby has 600 cents: 200 nickels (5 cents each), 200 dimes (10 cents each), and 200 quarters (25 cents each). Abby loves chips and wants to buy as many bags of chips as possible with the coins she has. She is buying from a vending machine that only accepts exact change, and the price of each bag of chips is 45 cents. How many bags of chips can she buy?

Solution:

Abby can purchase 150 burritos. For example, she can buy 50 of them with a quarter and four nickels, and 100 of them with a quarter and two dimes.

To show Abby cannot purchase more than 150 burritos, notice that she can only use one quarter to purchase each one, and the remaining 20 cents of each burrito must be purchased with nickels and dimes. Abby has 3000 cents worth of nickels and dimes, and $3000/20 = 150$, so she can purchase only up to 150 burritos.

- Prove or disprove this claim: it is possible to cross out one of the factors of the form $n!$ in the product $(100!)(99!)(98!)\dots(3!)(2!)(1!)$ so that the result is a perfect square.

Solution: It is indeed possible to do so. Let's first write the product as

$$\begin{aligned} (100!)(99!)\dots(2!)(1!) &= (100 \cdot 99 \dots 3 \cdot 2)(99 \cdot 98 \dots 3 \cdot 2)\dots(4 \cdot 3 \cdot 2)(3 \cdot 2)(2) \\ &= (2^{99})(3^{98})(4^{97})\dots(98^3)(99^2)(100) \end{aligned}$$

Every number raised to an even power is a perfect square. If we factor out one factor from each number raised to an odd power, we will get

$$\begin{aligned} (100!)(99!) \dots (2!)(1!) &= (2 \cdot 4 \cdot 6 \dots 98 \cdot 100)(2^{98})(3^{98})(4^{96}) \dots (98^2)(99^2) \\ &= (2 \cdot 4 \cdot 6 \dots 98 \cdot 100) \cdot A, \end{aligned}$$

with A being a perfect square because it is the product of even powers of integers. Now $(2 \cdot 4 \cdot 6 \dots 98 \cdot 100) = (2 \cdot 1)(2 \cdot 2)(2 \cdot 3) \dots (2 \cdot 49)(2 \cdot 50) = 2^{50}(50!)$ and 2^{50} is a perfect square. Thus, if we cross out $50!$ from the original product we get 2^{50} , which is a perfect square. \square

4. Prove that

$$1 - \cot 23 = \frac{2}{1 - \cot 22}.$$

Solution:

It is key to recognize that $\cot(45) = 1$. From this, we can derive

$$\begin{aligned} 1 &= \cot(45) \\ &= \cot(23 + 22) \\ &= \frac{(\cot(23))(\cot(22)) - 1}{\cot 23 + \cot 22} \end{aligned}$$

This implies that

$$\cot 23 + \cot 22 = (\cot(23))(\cot(22)) - 1,$$

which we can treat as a Diophantine equation and use Simon's Favorite Factoring Trick.

$$\begin{aligned} (\cot(23))(\cot(22)) - \cot 23 - \cot 22 - 1 &= 0 \\ (1 - \cot(23))(1 - \cot(22)) - 2 &= 0 \end{aligned}$$

Therefore,

$$1 - \cot 23 = \frac{2}{1 - \cot 22}. \quad \square$$

5. Determine the largest positive integer that is a factor of $n^4 \cdot (n - 1)^3 \cdot (n - 2)^2 \cdot (n - 3)$ for all positive integers n.

Solution 1:

Let $f(n) = n^2(n-1)^3(n-2)^2(n-3)$, and let d be the greatest positive integer that divides $f(n)$ for all positive integers n . Then in particular, d divides each of the numbers

$$\begin{aligned} f(4) &= 4^4 \cdot 3^3 \cdot 2^2 \cdot 1 = 2^{10} \cdot 3^3, \\ f(5) &= 5^4 \cdot 4^3 \cdot 3^2 \cdot 2 = 2^7 \cdot 5^4, \\ f(7) &= 7^4 \cdot 6^3 \cdot 5^2 \cdot 4 = 2^5 \cdot 3^3 \cdot 5^2 \cdot 7^4. \end{aligned}$$

Therefore, d divides $\gcd(f(4), f(5), f(7)) = 2^5 \cdot 3^2 = 288$. We claim that $f(n)$ is divisible by 288 for all positive integers n .

If n is even, then $n-2$ is also even, so $f(n)$ is divisible by $2^4 \cdot 2^2 = 2^6$. If n is odd, then $n-1$ and $n-3$ are both even, and one of them is divisible by 4. In particular, n is at least divisible by $2^3 \cdot 4 = 2^5$. Hence, $f(n)$ is divisible by 2^5 for all positive integers n .

Also, for all positive integers n , one of the terms $n, n-1$, or $n-2$ must be divisible by 3, which implies that $f(n)$ is always divisible by 3^2 .

Therefore, $f(n)$ is divisible by $2^5 \cdot 3^2 = 288$ for all positive integers n . Thus, the answer is $\boxed{288}$.

Solution 2:

As in Solution 1, d must divide 288. But

$$n^4(n-1)^3(n-2)^2(n-3) = 288 \cdot \binom{n}{4} \cdot \binom{n}{3} \cdot \binom{n}{2} \cdot \binom{n}{1},$$

which is always divisible by 288, so the answer is $\boxed{288}$.