2016-17 Monthly Contest 3 Solutions

Note: The following solutions are not the only solutions possible. We encourage you to seek other solutions, and perhaps yours will be more elegant than ours!

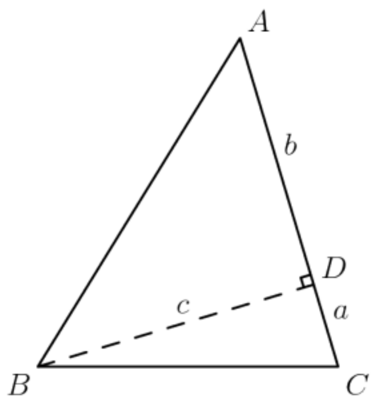
1. Xavier, Yvonne, and Zach play badminton. Two players play at a time, and the third person plays the winner of the previous game (ties are not possible). Eventually, Xavier played 10 games while Yvonne played 21 games. How many games did Zach play?

   Solution:

   We can see that Xavier must play at least once in every two games, because if he is not in the first game of the two, he will play the winner of the first game in the second game. Since Yvonne played at least 21 games, the total number of games must be greater than or equal to 21. However, if there were more than 21 games, Xavier must have played at least $22/2 = 11$ games, which means that there were a total of 21 games only. Yvonne would have then played all 21 games, ten of which were against Xavier, so Zach played $21 - 10 = \boxed{11}$ games.

2. The lengths of the sides of an acute angled triangle ABC are consecutive integers with $BC < CA < AB$. Let BD be the altitude from B to CA. Prove that $AD - CD = 4$.

   Solution:



   Let $BC = x - 1, CA = x$, and $AB = x + 1$. Then let $CD = a, DA = b$, and $BD = c$. By the Pythagorean theorem on triangle BCD, we have

$$c^2 + a^2 = (x - 1)^2.$$

By the Pythagorean theorem on triangle BDA, we have

$$c^2 + b^2 = (x+1)^2.$$

Subtracting these two equations, we have

$$b^2 - a^2 = (x+1)^2 - (x-1)^2 = 4x.$$

By difference of squares, $b^2 - a^2 = (b+a)(b-a)$. But $b + a = x$ by construction, so we have

$$x(b-a) = 4x,$$

or $b - a = 4$, as desired. □

3. (ARML 2016 Team, #9) Compute the number of permutations $x_1, x_2, ..., x_{10}$ of the integers $-3, -2, -1, ..., 6$ that satisfy the chain of inequalities $x_1x_2 \le x_2x_3 \le ... \le x_9x_{10}$.

   Solution (from the official ARML solution):

   First note that no two negative numbers can occur consecutively because otherwise, either the resulting positive product would have to be followed by a non-positive product once the negatives are exhausted, or it would have to occur at the end but there are too many positive terms for that to be possible. Thus the sequence must start with alternating positive and negative values. Each inequality $x_ix_{i+1} \le x_{i+1}x_{i+2}$ implies that $x_i < x_i + 2$ if $x_{i+1} > 0$ and $x_i > x_{i+2}$ if $x_{i+1} < 0$ (the inequalities are strict because the xis are distinct). The two main cases to consider are $x_1 = -3$ and $x_2 = -3$.

   If $x_1 = -3$, then $x_3 = -2$, $x_5 = -1$, and either $x_6 = 0$ or $x_7 = 0$. If $x_6 = 0$, then the positive entries (x_2, x_4, x_7, x_8, x_9, and x_{10}) must satisfy $x_2 > x_4$, $x_7 < x_9$, and $x_8 < x_{10}$. There are $\frac{6!}{2!\cdot 2!\cdot 2!} = 90$ such sequences. If $x_7 = 0$, then the positive entries (x_2, x_4, x_6, x_7, x_8, and x_{10}) must satisfy $x_2 > x_4 > x_6$, and $x_8 < x_{10}$ for $\frac{6!}{3!\cdot 2!\cdot 1!} = 60$ more.

   If $x_2 = -3$, then $x_4 = -2, x_6 = -1$, and either $x_7 = 0$ or $x_8 = 0$. If $x_7 = 0$, then the positive entries (x_1, x_3, x_4, x_8, x_9, and x_{10}) must satisfy $x_1 > x_3 > x_5$, and $x_8 < x_{10}$ for another 60. If $x_8 = 0$, then the positive entries (x_1, x_3, x_5, x_7, x_9, and x_{10}) must satisfy $x_1 > x_3 > x_5 > x_7$, for $\frac{6!}{4!\cdot 1!\cdot 1!} = 30$ more. The desired number of permutations is therefore $90 + 60 + 60 + 30 = \boxed{240}$.

4. Let $A_1A_2A_3...A_8$ be a regular octagon with O as its center. Triangular regions OA_iA_{i+1}, $1 \le i \le 8 (A_9 = A_1)$ are to be colored red, blue, and

green such that adjacent regions are colored in different colors. In how many ways can this be done?

Solution:
(Problem based from 102 Combinatorical Problems)

We'll begin by trying to find the general formula. Let $A_1A_2A_3...A_8$ ($n \geq 3$) be a regular n-sided polygon with O as its center. Triangular regions OA_iA_{i+1}, $1 \leq i \leq n$ (and $A_{n+1} = A_1$ are to be colored in one of the k (k $\geq$ 3) colors such that the adjacent regions are different colors. Let $P_{n,k}$ denote the number of such colorings. We want to find $P_{8,3}$.

There are k ways to ways to color the region OA_1A_2, and then k-1 ways to color regions OA_2A_3, OA_3A_4,and so on. We have to be careful about the coloring of the region OA_nA_1 because it is possible that it has the same coloring as OA_1A_2. However, if this is the case, then we simply end up with a legal coloring for n-1 regions by viewing region OA_nA_2 as one region. This is a clear bijection between this special kind of illegal colorings of n regions to legal colorings of n-1 regions. Thus, $P_{n,k} = k(k-1)^{n-1} - P_{n-1,k}$. Note that $P_{3,k} = k(k-1)(k-2)$

$$\begin{aligned}
P_{n,k} &= k(k-1)^{n-1} - k(k-1)^{n-2} + k(k-1)^{n-3} - ... \\
&+ (-1)^{n-4}k(k-1)^3 + (-1)^{n-3}k(k-1)(k-2) \\
&= k\frac{(k-1)^n + (-1)^{n-4}(k-1)^3}{1+(k-1)} + (-1)^{n-3}(k-1)(k-2) \\
&= (k-1)^n + (-1)^{n-4}(k-1)^3 + (-1)^{n-1}(k-1)(k-2) \\
&= (k-1)^n + (-1)^n(k-1)[(k-1)^2 - k(k-2)] \\
&= (k-1)^n + (-1)^n(k-1).
\end{aligned}$$

So $P_{8,3} = 2^8 + (-1)^8(2) = 256 + 2 = \boxed{258}$ ways to color this octagon.

5. Given a 20 by 20 board, prove whether or not it is possible to fill with 2 by 1 dominoes when 2 opposing corners of the board are removed.

   Solution:

   No matter how a domino is placed, a 2 by 1 domino will always cover exactly 1 black square and 1 white square on a board. Note that because the colors alternate on a board, opposing corners will always have the same color. WLOG, we can assume that the 2 opposing corners being removed are black. Now, we are left with 200 white squares and $200 - 2 = 198$ black squares. Since the number of black and white squares are not equal, and because every domino needs to cover a black square and a white square, it is not possible to fill the board with 2 by 1 dominoes. ☐