## 2016-17 Monthly Contest 1 Solutions

Note: The following solutions are not the only solutions possible. We encourage you to seek other solutions, and perhaps yours will be more elegant than ours!

1. Given that two real numbers $x$ and $y$ satisfy the two equations $x-y=4$ and $xy=2$, find all possible values of $x^5-y^5$.

   **Solution:** Many students solved this question by factoring $x^5-y^5$. While this works, it is quite tedious. An easier solution is provided below. This question is an excellent example of choosing between the elegance of factoring and the efficiency of brute force.

   From the first equation, we get that $x=4+y$. Substituting this into the second equation, we get the quadratic

   $$(4+y)y=2$$
   $$y^2+4y-2=0$$

   Solving the quadratic for $y$ and substituting the values of $y$ back into $x=4+y$ to get $x$, we retrieve the two pairs of solutions

   $$(x,y)=(-\sqrt{6}+2,-\sqrt{6}-2),(\sqrt{6}+2,\sqrt{6}-2).$$

   Substituting both pairs of $x$ and $y$ into $x^5-y^5$ returns the same value of $\boxed{1744}$.

2. Three dice are rolled. What is the probability that the sum of the rolls is 8?

   **Solution:** All the unordered rolls that sum to 8 on three dice are $(1,1,6),(1,2,5),(1,3,4),(2,2,4)$, and $(2,3,3)$.

   - There are $\frac{3!}{2!}=3$ permutations of $(1,1,6)$.
   - There are $3!=6$ permutations of $(1,2,5)$.
   - There are $3!=6$ permutations of $(1,3,4)$.
   - There are $\frac{3!}{2!}=3$ permutations of $(2,2,4)$.
   - There are $\frac{3!}{2!}=3$ permutations of $(2,3,3)$.

There is a total of $3+6+6+3+3 = 21$ ordered rolls that sum to 8 on three dice. There are $6^3 = 216$ total possibilities from rolling 3 dice, so the probability is

$$\frac{21}{216} = \boxed{\frac{7}{72}}.$$

3. A triangle ABC has side lengths AB = 10, AC =12, and BC =17. There is a point D such that BD = 7 and DC = 10. Find AD.

   **Solution:** By Stewart's Theorem (for more info, click here) , we get the equation

$$AC^2 \cdot BD + AB^2 \cdot DC = BD \cdot DC \cdot BC + AD^2 \cdot BC$$
$$12^2 \cdot 7 + 10^2 \cdot 10 = 7 \cdot 10 \cdot 17 + AD^2 \cdot 17$$

   Solving for AD yields AD = $\boxed{\sqrt{\frac{818}{17}}}$.

4. Bob thinks of a number between 1 and 32. Alice can ask Bob any questions she wants to, as long as the answer can be yes or no. How can Alice figure out Bobs number with at most five questions?

   **Solution:** This question essentially tests the understanding of binary search. The same solution can be applied if all the questions asked "Is your number less than ...?"

   Let Alice's first question be "Is your number bigger than 16?" If the answer is no, Bob's number is 1 of 16 numbers: $1, 2, 3, ..., 16$. If the answer is yes, the number is 1 of the other 16 numbers: $17, 18, 19, ..., 32$.

   On the next question, Alice can again cut the possible numbers in half. She should ask "Is your number bigger than 8?" if Bob answered no to the first question, and "Is your number bigger than 24?" if Bob answered yes to the first question.

   Continuing in the same manner, Alice will cut in half the number of possible choices for Bob's number. Since $32 = 2^5$, Alice will know Bob's number after the fifth question.

5. Prove that for all natural numbers $x$ and $n$, $(1 + x)^n \geq 1 + nx$.

   **Solution:** This inequality is actually known as Bernoulli's inequality, if you want to learn more about it! This proof is best done by induction.

   Let $n = 1$. Then $1 + x \geq 1 + x$, which always holds true. Thus, the base case is true.

Now assume a typical case $k$ and show that the next case $k+1$ follows. That is, assume

$$(1+x)^k \geq 1 + kx, \tag{1}$$

and derive

$$(1+x)^{k+1} \geq 1 + (k+1)x. \tag{2}$$

To derive (2), start with its lefthand side and by substituting (1) and algebra, get its righthand side:

$$\begin{aligned} (1+x)^{k+1} = (1+x)^k \cdot (1+x) &\geq (1+kx)(1+x) \\ &= 1 + kx + x + kx^2 \\ &= 1 + (k+1)x + kx^2 \\ &= 1 + (k+1)x. \ \square \end{aligned}$$