Monthly Contest 5
Due March 22, 2017

Instructions

This contest consists of 5 problems, each worth 7 points. Problems may not be of equal difficulty. Please write your solution to every problem on a separate sheet of paper. If you use more than one sheet for a specific problem, please staple the sheets together. Do NOT staple together solutions to different problems. It would be preferred if you used 8.5 by 11 printer paper to write your solutions on and leave a small margin around the edges. On the top of each sheet, include the contest number, problem number, and page number and the total number of pages of the problems solution (not the total pages of all the solutions).

DO NOT put your name anywhere on your solutions. Doing so will result in instant disqualification. When you turn your solutions in, you will be asked to write a student code (which will be provided on the due date) on your solutions.

You must justify all answers to receive full credit. Answers without justification will receive few, if any, points. Partial credit will be given for significant progress into solving a problem. Leave answers in exact form unless otherwise specified. Do not feel discouraged if you cannot solve all the problems. It is already a great accomplishment if you are able to solve one of these problems. You are NOT allowed to consult or talk to anyone else about the problems, whether in person, on the phone, via e-mail, on forums, or any other means of communication. You may view any book or website (including forums) as long as you do not communicate the problem in any sort of manner (e.g. posting the problem statement or variations on a forum).

Problems

1. Let n be a positive integer. Prove that $5n^2 + 4$ or $5n^2 - 4$ is a perfect square if and only if n is a Fibonacci number.

2. Let P be an interior point of ΔABC, and extend lines from the vertices through P to the opposite sides. Let AP = a, BP = b, CP = c, and let the extentions from P to the opposite sides all have length d. If $a + b + c = 40$ and $d = 4$, find abc.

3. Two spies, Alex and Bethany, must pass each other their secret IDs, which are integers in the range 1-1700. They meet at a river, where there is a pile of 26 indistinguishable stones. Starting with Alex, they take turns throwing a group of stones into the river. Each spy much

throw at least one stone on his/her turn, until all the stones are gone. They observe all throws and leave when there are no more stones. No information is exchanged except the number of stones thrown at each turn. How can they exchange the numbers successfully?

4. For how many pairs of consecutive integers in the set

$$\{1000, 1001, 1002, ..., 2000\}$$

is no carrying required when the two integers are added?

5. Prove that for any integer $n > 1$

$$n! < (\frac{n+1}{2})^n$$

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