Monthly Contest 4
Due February 22, 2017

Instructions

This contest consists of 5 problems, each worth 7 points. Problems may not be of equal difficulty. Please write your solution to every problem on a separate sheet of paper. If you use more than one sheet for a specific problem, please staple the sheets together. Do NOT staple together solutions to different problems. It would be preferred if you used 8.5 by 11 printer paper to write your solutions on and leave a small margin around the edges. On the top of each sheet, include the contest number, problem number, and page number and the total number of pages of the problems solution (not the total pages of all the solutions).

DO NOT put your name anywhere on your solutions. Doing so will result in instant disqualification. When you turn your solutions in, you will be asked to write a student code (which will be provided on the due date) on your solutions.

You must justify all answers to receive full credit. Answers without justification will receive few, if any, points. Partial credit will be given for significant progress into solving a problem. Leave answers in exact form unless otherwise specified. Do not feel discouraged if you cannot solve all the problems. It is already a great accomplishment if you are able to solve one of these problems. You are NOT allowed to consult or talk to anyone else about the problems, whether in person, on the phone, via e-mail, on forums, or any other means of communication. You may view any book or website (including forums) as long as you do not communicate the problem in any sort of manner (e.g. posting the problem statement or variations on a forum).

Problems

1. Let n be an odd integer greater than 1. Prove that the sequence $\binom{n}{1},\binom{n}{3},\binom{n}{5}, \ldots, \binom{n}{(n-2)}$ contains an odd number of odd numbers.

2. Abby has 600 coins: 200 nickels (5 cents each), 200 dimes (10 cents each), and 200 quarters (25 cents each). Abby loves chips and wants to buy as many bags of chips as possible with the coins she has. She is buying from a vending machine that only accepts exact change, and the price of each bag of chips is 45 cents. How many bags of chips can she buy?

3. Prove or disprove this claim: it is possible to cross out one of the factors of the form n! in the product (100!)(99!)(98!)...(3!)(2!)(1!) so that the result is a perfect square.

4. Prove that
$$1 - \cot 23^\circ = \frac{2}{1 - \cot 22^\circ}$$

5. Determine the largest positive integer that is a factor of $n^4 \cdot (n-1)^3 \cdot (n-2)^2 \cdot (n-3)$ for all positive integers n.