SUMS AND PRODUCTS

1. *(1999 Invitational World Youth Mathematics Intercity Competition, Taiwan, Individual Contest, §5)*
   Calculate
   \[1999^2 - 1998^2 + 1997^2 - 1996^2 + \cdots + 3^2 - 2^2 + 1^2.\]

2. *(1979 AMC 12, §11)* Find a positive integral solution to the equation:
   \[
   \frac{1 + 3 + 5 + \cdots + (2n - 1)}{2 + 4 + 6 + \cdots + (2n)} = \frac{115}{116}.
   \]
   (A) 110 (B) 115 (C) 116 (D) 231 (E) the equation has no positive integral solutions

3. *(2005 Archimedes Contest, Romania, Grade 6, Final Round, §2)* There exists a positive integer \(n\) such that
   \[
   \frac{8000}{19} \left( \frac{1}{1 + 2 + 3 + \cdots + 100} + \frac{1}{1 + 2 + 3 + \cdots + 101} + \cdots + \frac{1}{1 + 2 + 3 + \cdots + 1999} \right) = n^3.
   \]
   Find \(n\).

4. *(2012 Exeter Math Club Competition, Individual Accuracy Test, §9)* Let \(f(x) = \sqrt{2x + 1} + 2\sqrt{x^2 + x}\).
   Determine the value of
   \[
   \frac{1}{f(1)} + \frac{1}{f(2)} + \frac{1}{f(3)} + \cdots + \frac{1}{f(24)}.
   \]

5. *(1977 AMC 12, §24)* Find the sum \(\frac{1}{1 \cdot 3} + \frac{1}{3 \cdot 5} + \cdots + \frac{1}{(2n - 1)(2n + 1)} + \cdots + \frac{1}{255 \cdot 257}\).
   (A) \(\frac{127}{255}\) (B) \(\frac{128}{255}\) (C) \(\frac{1}{2}\) (D) \(\frac{128}{257}\) (E) \(\frac{129}{257}\)

6. Calculate:
   \[
   \frac{3^2 + 1}{3^2 - 1} + \frac{5^2 + 1}{5^2 - 1} + \frac{7^2 + 1}{7^2 - 1} + \cdots + \frac{99^2 + 1}{99^2 - 1}.
   \]

7. Find the sum:
   \[
   \frac{1}{1 + 1^2 + 1^4} + \frac{2}{1 + 2^2 + 2^4} + \frac{3}{1 + 3^2 + 3^4} + \cdots + \frac{50}{1 + 50^2 + 50^4}.
   \]

8. *(2001-2002 Mandelbrot Competition, Round 2 Individual, §7)* Define a sequence of numbers by
   \[a_n = 3n^2 + 3n + 1,\]
   so that \(a_1 = 7, a_2 = 19, a_3 = 37,\) and so on. Calculate \(a_1 + a_2 + \cdots + a_{100}\).
9. \((1995-1996 \text{ Mandelbrot Competition, Round 2 Individual, \#7})\) A sequence \(a_1, a_2, a_3, \ldots\) is defined recursively by \(a_1 = 1, a_2 = 1, \) and for \(k \geq 3,\)
\[ a_k = \frac{1}{3} a_{k-1} + \frac{1}{4} a_{k-2}. \]
Evaluate \(a_1 + a_2 + a_3 + \ldots\)

10. \((1996 \text{ Turkey Math Contest})\) Let \(a_n\) be the integer closest to \(\sqrt{n}.\) Find the sum
\[ S = \frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_{2070}}. \]

11. \((2002 \text{ AIME I, \#4})\) Consider the sequence defined by \(a_k = \frac{1}{k^2 + k}\) for \(k \geq 1.\) Given that
\[ a_m + a_{m+1} + \cdots + a_{n-1} = \frac{1}{29} \]
for positive integers \(m\) and \(n\) with \(m < n,\) find \(m + n.\)

12. \((1992-1993 \text{ Mandelbrot Competition, Round 5 Individual, \#5})\) Determine the value of the infinite sum
\[ \sum_{n=17}^{\infty} \binom{n}{15} \binom{n}{17}. \]

13. \((2002 \text{ AIME II, \#6})\) Find the integer that is closest to \(1000 \sum_{n=3}^{10000} \frac{1}{n^2 - 4}.\)

14. \((1972 \text{ AMC 12, \#19})\) The sum of the first \(n\) terms of the sequence
\[ 1, (1 + 2), (1 + 2 + 2^2), \ldots, (1 + 2 + 2^2 + \cdots + 2^{n-1}) \]
in terms of \(n:\)
(A) \(2^n\)  (B) \(2^n - n\)  (C) \(2^{n+1} - n\)  (D) \(2^{n+1} - n - 2\)  (E) \(n \cdot 2^n\)

15. \((2006 \text{ Harvard-MIT Math Tournament, Algebra, \#4})\) Let \(a_1, a_2, \ldots\) be a sequence defined by \(a_1 = a_2 = 1\)
and \(a_{n+2} = a_{n+1} + a_n\) for \(n \geq 1.\) Find \(\sum_{n=1}^{\infty} \frac{a_n}{4^n}.\)

16. \((2001-2002 \text{ Mandelbrot Competition, Round 4 Individual, \#7})\) Let \(F_n\) be the \(n^{\text{th}}\) Fibonacci number, where as usual \(F_1 = F_2 = 1\) and \(F_{n+1} = F_n + F_{n-1}\) for all \(n \geq 2.\) Find the value of the infinite sum
\[ \frac{1}{3} + \frac{1}{9} + \frac{2}{27} + \cdots + \frac{F_n}{3^n} + \ldots \]
17. (1994-1995 Mandelbrot Competition, Round 1 Individual, #5) It is known that \( \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots = \frac{\pi^2}{6} \). Given this fact, determine the exact value of
\[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots.\]

18. (1998-1999 Mandelbrot Competition, Round 1 Individual, #8) Define \( \tau(n) = 0 \) if the highest power of 2 dividing \( n \) is odd, and \( \tau(n) = 1 \) if the highest power of 2 dividing \( n \) is even. For example, \( \tau(4) = 1 \) and \( \tau(5) = 1 \), but \( \tau(6) = 0 \). Compute \( \sum_{n=1}^{\infty} \frac{\tau(n)}{n^2} \).

19. (2009 AMC 12 A, #17) Let \( a + ar + ar^2 + ar^3 + \cdots \) and \( a + ar_2 + ar_2^2 + ar_2^3 + \cdots \) be two different infinite geometric series of positive numbers with the same first term. The sum of the first series is \( r_1 \), and the sum of the second series is \( r_2 \). What is \( r_1 + r_2 \)?

(A) 0 (B) \( \frac{1}{2} \) (C) 1 (D) \( \frac{1 + \sqrt{5}}{2} \) (E) 2

20. (1962 AMC 12, #40) The limiting sum of the infinite series \( \frac{1}{10} + \frac{2}{10^2} + \frac{3}{10^3} + \cdots \) whose \( n^{\text{th}} \) term is \( \frac{n}{10^n} \) is:

(A) \( \frac{1}{9} \) (B) \( \frac{10}{81} \) (C) \( \frac{1}{8} \) (D) \( \frac{17}{72} \) (E) larger than any finite quantity

21. (1979 IMO, #1) Let \( p \) and \( q \) be positive integers such that
\[\frac{p}{q} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots - \frac{1}{1318} + \frac{1}{1319}.\]
Prove that \( p \) is divisible by 1979.

22. (1998 Gazeta Matematica, Romania) Let
\[A = \frac{1}{1 \cdot 2} + \frac{1}{3 \cdot 4} + \cdots + \frac{1}{2011 \cdot 2012}\]
and
\[B = \frac{1}{1007 \cdot 2012} + \frac{1}{1008 \cdot 2011} + \cdots + \frac{1}{2012 \cdot 1007}.\]
Evaluate \( A/B \).

23. (2011 San Jose State University, Problem of the Week) Let \( p \) and \( q \) be positive integers such that:
\[\frac{p}{q} = 1 + \frac{1}{2} - \frac{2}{3} + \frac{4}{5} - \frac{1}{6} + \frac{1}{7} - \frac{2}{9} + \frac{1}{10} + \cdots + \frac{1}{1505} - \frac{2}{1506} + \frac{1}{1507} + \frac{1}{1508}.\]
Prove that \( p \) is divisible by 2011.

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24. \((1989 \text{ AMC 12, } \#29)\) Find \(\sum_{k=0}^{49} (-1)^k \binom{99}{2k}\), where \(\binom{n}{j} = \frac{n!}{j!(n-j)!}\).

(A) \(-2^{50}\) (B) \(-2^{49}\) (C) 0 (D) \(2^{49}\) (E) \(2^{50}\)

25. Evaluate: \((4 \times 7 + 2)(6 \times 9 + 2)(8 \times 11 + 2) \ldots (100 \times 103 + 2)\)
\((5 \times 8 + 2)(7 \times 10 + 2)(9 \times 12 + 2) \ldots (99 \times 102 + 2)\).

26. \((1976 \text{ AMC 12, } \#21)\) What is the smallest positive odd integer \(n\) such that the product \(2^{1/7}2^{3/7} \ldots 2^{(2n+1)/7}\) is greater than 1000? (In the product the denominators of the exponents are all sevens, and the numerators are the successive odd integers from 1 to \(2n + 1\).)

(A) 7 (B) 9 (C) 11 (D) 17 (E) 19

27. \((1991 \text{ AMC 12, } \#25)\) If \(T_n = 1 + 2 + 3 + \ldots + n\) and \(P_n = \frac{T_2}{T_2 - 1} \cdot \frac{T_3}{T_3 - 1} \cdot \frac{T_4}{T_4 - 1} \ldots \frac{T_n}{T_n - 1}\) for \(n = 2, 3, 4, \ldots\), then \(P_{1991}\) is closest to which of the following numbers?

(A) 2.0 (B) 2.3 (C) 2.6 (D) 2.9 (E) 3.2

28. \((1997-1998 \text{ Mandelbrot Competition, Round 4 Individual, } \#4)\) Compute the product:
\[
\]

29. \((1971 \text{ AMC 12, } \#32)\) If \(s = (1 + 2^{-1/32})(1 + 2^{-1/16})(1 + 2^{-1/8})(1 + 2^{-1/4})(1 + 2^{-1/2})\), then \(s\) is equal to:

(A) \(\frac{1}{2}(1 - 2^{-1/32})^{-1}\) (B) \((1 - 2^{-1/32})^{-1}\) (C) \(1 - 2^{-1/32}\) (D) \(\frac{1}{2}(1 - 2^{-1/32})\) (E) \(\frac{1}{2}\)

30. \((2005 \text{ AIME II, } \#7)\) Let \(x = \frac{4}{(\sqrt{5} + 1)(\sqrt{5} + 1)(\sqrt{5} + 1)(\sqrt{5} + 1)}\). Find \((x + 1)^{48}\).

31. \((1987 \text{ AIME, } \#14)\) Compute:
\[
\]
32. (2001-2002 Mandelbrot Competition, Round 3 Individual, #6) Let $F_n$ be the $n$th Fibonacci number, where as usual $F_1 = F_2 = 1$ and $F_{n+1} = F_n + F_{n-1}$. Find the value of the product

$$\prod_{k=2}^{100} \left( \frac{F_k}{F_{k-1}} - \frac{F_k}{F_{k+1}} \right),$$

leaving your answer in terms of exactly two Fibonacci numbers.

33. (2004 AMC 12 A, #25) For each integer $n \geq 4$, let $a_n$ denote the base-$n$ number $0.\overline{133}_n$. The product $a_4a_5\ldots a_{99}$ can be expressed as $\frac{m}{n!}$, where $m$ and $n$ are positive integers and $n$ is as small as possible. What is the value of $m$?

(A) 98  (B) 101  (C) 132  (D) 798  (E) 962