POWER ROUND

7th Polya Mathematics Competition 30 October 1999

Please submit written proofs in legible and coherent style. Even where a problem calls for finding solutions instead of proving, be sure you give some argument that convinces the reader you have found all possible solutions. Write the team name and problem number clearly at the top of each sheet of paper you submit. Please don’t combine two different numbered problems on the same sheet, though lettered sub-problems may be solved on the same sheet if you wish. Each part has a point value indicated in [ ]. Partial credit will be awarded for partial solutions.

Your team has one hour to work together to solve these problems. Please spend some of that time on proofreading and checking each other’s work! When the hour is up, before you put your answers in the envelope, please number each page sequentially and indicate the total number of pages you have used. Make sure your team name is written on the outside of the envelope.

Whether or not you solve a part, you may still use the result to aid in the solution of later parts. The problems proceed roughly in order of difficulty, but some of the later parts may still be easier than some of the earlier ones, so make sure your team looks at all the problems!

No calculators or other aids are allowed.

This round explores nonnegative integer solutions \((x, y)\) to equations of the form \(x^2 - dy^2 = n\). Remember, \(d\) and/or \(n\) may be any integers (positive, negative, or 0), but we are only interested in nonnegative solutions \((x, y)\). This equation is called Pell’s equation.

1. [2] Prove that \(x^2 - dy^2 = 0\) can be solved if and only if \(d\) is a perfect square.

2. (a) [1] Find all solutions to \(x^2 - 4y^2 = 1\).
   
   (b) [2] Prove that, if \(d\) is a perfect square, then \(x^2 - dy^2 = 1\) has only finitely many solutions.
   
   (c) [1] Prove that, if \(d\) is a perfect square, then \(x^2 - dy^2 = n\) has only finitely many solutions.

3. (a) [1] Find all solutions to \(x^2 + 3y^2 = 27\).
   
   (b) [2] Prove that, if \(d\) is negative, \(x^2 - dy^2 = n\) has only finitely many solutions.
   
   (c) [1] Give an example of values of \(d < 0\) and \(n > 100\) such that \(x^2 - dy^2 = n\) has no integer solutions \((x, y)\). (Be sure to explain how you know it has no solutions!)

4. This problem aims at finding all possible solutions to \(x^2 - 2y^2 = 1\).
   (a) [2] Find the two smallest solutions \((x_0, y_0)\) and \((x_1, y_1)\). [Hint: guess-and-check, and justify your answer by showing that you guessed every possible number.]
   
   (b) [2] The next larger solution after those two is \((x_2, y_2) = (17, 12)\). There’s a simple linear relationship that generates each of these solutions from the previous one. That is, \((x_{k+1}, y_{k+1}) = (rx_k + sy_k, tx_k + uy_k)\). Using your answer to (a), find the values of \(r, s, t,\) and \(u\).
   
   (c) [2] Prove that the formula you found in (b) always produces solutions.

(continued on back)
5. (a) [1] Compute \((3 + 2\sqrt{2})^2\). [Showing your algebra steps is enough proof for this problem].

(b) [2] How does \((3 + 2\sqrt{2})^k\) relate to your answer to problem 4 (parts b and/or c)? Prove the relationship.

6. (a) [1] Find the two smallest solutions to \(x^2 - 8y^2 = 1\).

(b) [1] Find the next larger solution.

(c) [2] Find and prove a linear relationship as in problem 4b that produces further solutions.

7. For each of the following equations:
   If there are no solutions, prove it.
   If there are a non-zero but finite number of solutions, prove it, and give at least one solution.
   If there are infinitely many solutions, find and prove a formula (as in problems 4b and 4c) that will generate infinitely many solutions. Give at least three solutions.
   (a) [2] \(x^2 - 2y^2 = -1\).

(b) [2] \(x^2 - 3y^2 = -1\).

8. By now you should understand that solutions to \(x^2 - 2y^2 = 1\) has a lot to do with the square root of 2. This problem explores that connection, through an expression called a "continued fraction".
   One way to compute a good rational approximation to a number is to take the integer part. Then what's left over is between 0 and 1, so take its reciprocal, and then take the integer part of that. For example,
   \[
   \sqrt{2} = 1 + \frac{1}{\sqrt{2} - 1}
   \]
   \[
   \frac{1}{\sqrt{2} - 1} = \sqrt{2} + 1 = 2 + (\sqrt{2} - 1)
   \]
   shows that \(\sqrt{2} = 1 + (\text{something left over})\), so the first approximation is 1.
   \[
   \sqrt{2} = 1 + \frac{1}{2 + (\text{something left over})}
   \]
   so the second approximation is \(1 + 1/2 = 3/2\).

(a) [1] Continue this process (compute an approximation to \(1/(\text{something left over})\)) to find the third approximation. (show work!)

(b) [1] Explain (you need not prove!) the relationship between this process and problems 4b and 7a.

(c) [1] Prove that relationship.

(d) [1] In the unlikely event that you have enough time left to make this problem worthwhile for just one point, repeat this process for the square root of 3 and relate it to the equations \(x^2 - 3y^2 = 1\) and \(x^2 - 3y^2 = -1\).

(e) [1] Generalize the results of this problem to \(x^2 - dy^2 = \pm 1\). When will infinitely many solutions exist?