Number Theory I:
Finding Fractions in an Infinite Tree

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The Calkin-Wilf tree is formed by beginning with the fraction $\frac{1}{1}$.

For any fraction $\frac{a}{b}$, its mother (on the left) is $\frac{a}{a+b}$ and its father is $\frac{a+b}{b}$. You can imagine them getting together because they have $a+b$ in common, and passing on the genetics of the numerator $a$ and the denominator $b$ to their child.

We number the rows (or generations), beginning with generation 0 for the fraction $\frac{1}{1}$.

We also number the fractions, beginning with number 1 for $\frac{1}{1}$.

1. If you follow the sequence MMDDDDMMMM, what fraction have you found?
2. What sequence should you follow to reach the fraction $\frac{17}{61}$?
3. What number does the fraction $\frac{17}{61}$ get?
4. What fraction is number 1000?
5. What is the product of all the fractions in the $n^{th}$ row?
6. What is the sum of all the fractions in the $n^{th}$ row?
7. What is the largest fraction in the $n^{th}$ row? How about the second largest?
8. What else do you notice? What do you wonder? What observations, patterns, conjectures, or questions can we find?
9. You might notice that the first 1 appears in row 1. Then in row 2 we find 2, and in row 3 we find 3. In row 4 we find 4 and 5. In row 5 we find 7 and 8. What happened to 6? Why is it "late"? Can we predict which numbers will appear particularly early or particularly late in the sequence? Can we figure out where in the list the number 1000 appears, or what the 1000$^{th}$ number in the list will be?