 Monthly Contest 5 Solutions

Here are the solutions for the fifth and last monthly contest of the 2013-2014 school year. The solutions listed below are not the only solutions. I encourage you to explore each problem and see if you can discover other approaches to the problems. Perhaps you may even find a solution more elegant than the ones listed!

**Problem 1** Let $a_1$, $a_2$, and $a_3$ be positive numbers such that $a_1 + a_2 + a_3 = 1$. Show that $a_1 a_2 a_3 \leq \frac{1}{27}$ and find when equality occurs.

**Solution** From AM-GM, we know that $\frac{1}{3} = \frac{a_1 + a_2 + a_3}{3} \geq \sqrt[3]{a_1 a_2 a_3}$. Cubing both sides, we get that

$$\frac{1}{27} = \left(\frac{1}{3}\right)^3 \geq (\sqrt[3]{a_1 a_2 a_3})^3 = a_1 a_2 a_3$$

**Problem 2** Show that $(1 + 2\sqrt{2} + 3\sqrt{3} + \cdots + (n - 1)\sqrt{n - 1} + n\sqrt{n})^2 \leq \left(\frac{n(n + 1)}{2}\right)\left(\frac{n(n + 1)(2n + 1)}{6}\right)$ for any positive integer $n$. Find when equality occurs.

**Solution** Cauchy-Schwartz gives us that

$$(1^2 + 2^2 + \cdots + n^2)((\sqrt{1})^2 + (\sqrt{2})^2 + \cdots + (\sqrt{n})^2) \geq (1 + 2\sqrt{2} + \cdots + n\sqrt{n})^2$$

Using the fact that $1 + 2 + \cdots + n = \frac{n(n + 1)}{2}$ and $1^2 + 2^2 + \cdots + n^2 = \frac{n(n + 1)(2n + 1)}{6}$, we get that

$$(1 + 2\sqrt{2} + 3\sqrt{3} + \cdots + (n - 1)\sqrt{n - 1} + n\sqrt{n})^2 \leq \left(\frac{n(n + 1)}{2}\right)\left(\frac{n(n + 1)(2n + 1)}{6}\right)$$

**Problem 3** Prove the Harmonic Mean-Geometric Mean (HM-GM) Inequality. That is, show that for positive numbers $a_1$, $a_2$, $a_3$, $\cdots$, $a_{n-1}$, $a_n$, the following holds true:

$$\frac{1}{\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} + \cdots + \frac{1}{a_{n-1}} + \frac{1}{a_n}} \leq \frac{1}{\sqrt[1]{a_1 a_2 a_3 \cdots a_{n-1} a_n}}$$

Also find when equality occurs.
Solution Note that by AM-GM, we have
\[ \frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n} \geq \sqrt[n]{\frac{1}{a_1} \cdot \frac{1}{a_2} \cdots \frac{1}{a_{n-1}} \cdot \frac{1}{a_n}} \]

By taking the reciprocal of both sides, we get
\[ \frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} + \cdots + \frac{1}{a_{n-1}} + \frac{1}{a_n} \leq \sqrt[n]{a_1a_2a_3\cdots a_{n-1}a_n} \]

Problem 4 Let \( p \) represent the perimeter of \( \triangle ABC \) and let \( R \) and \( S \) represent the circumradius and area, respectively. Prove that \( p^3 \geq 108RS \) and find when equality occurs.

Solution Let \( a, b, \) and \( c \) represent the side lengths of the triangle. Then by AM-GM, \( p = a + b + c \geq 3\sqrt[3]{abc} \). We can cube both sides to get \( p^3 \geq 27abc \). We also know that \( 4RS = abc \) so by combining these, we get \( p^3 \geq 27abc = 27(4RS) = 108RS \).

Problem 5 Prove that for nonnegative numbers \( a_1, a_2, a_3, \ldots, a_{n-1}, a_n \),
\[ a_1^2 + a_2^2 + a_3^2 + \cdots + a_{n-1}^2 + a_n^2 \geq (a_1 + a_2 + a_3 + \cdots + a_{n-1} + a_n) \sqrt[2n]{a_1a_2a_3\cdots a_{n-1}a_n} \]
and find when equality occurs.

Solution By Cauchy-Schwartz, we have that
\[ (a_1^2 + a_2^2 + \cdots + a_n^2) \left(1^2 + 1^2 + \cdots + 1^2\right) \geq (a_1 + a_2 + \cdots + a_n)^2 \]
Repeated \( n \) times

By rearranging and using AM-GM, we get
\[ a_1^2 + a_2^2 + \cdots + a_n^2 \geq (a_1 + a_2 + \cdots + a_n) \cdot \frac{(a_1 + a_2 + \cdots + a_n)}{n} \geq (a_1 + a_2 + \cdots + a_n) \sqrt[2n]{a_1a_2\cdots a_n} \]