Monthly Contest 1 Solutions

Here are the solutions for the first monthly contest of the 2013-2014 school year. The solutions listed below are not the only solutions. I encourage you to explore each problem and see if you can discover other approaches to the problems. Perhaps you may even find a solution more elegant than the ones listed!

Problem 1 Show that if we pick five lattice points (points \((x,y)\) on the coordinate plane such that \(x\) and \(y\) are integers), two of them must have a midpoint which is also a lattice point.

Solution The formula for the midpoint of two points \((x_1,y_1)\) and \((x_2,y_2)\) is \(\left(\frac{x_1+x_2}{2},\frac{y_1+y_2}{2}\right)\). In order for the midpoint to be integer, both \(x_1\) and \(x_2\) must be of the same parity and both \(y_1\) and \(y_2\) must be of the same parity. We may consider the following possible pairs of parity of the \(x, y\) coordinates: (even,even), (even,odd), (odd,even), and (odd,odd). We may note that if two of the points have the pair, then we are done as the sum of their \(x\) and \(y\) coordinates are both even and the midpoint will have integer coordinates. Since there are 5 points and 4 possible pairs, the Pigeonhole Principle states that two points must belong to the same pair and two points have a lattice point for their midpoint.

Problem 2 Let the sequence of integers \(a_0,a_1,a_2,\ldots,a_n\) be called interesting if it has the property that for every integer \(0 \leq k \leq n\), the number of times \(k\) appears in the sequence is \(a_k\). For example, \(a_0 = 1, a_1 = 2, a_2 = 1, a_3 = 0\) would be called an interesting sequence. Given an interesting sequence \(a_0,a_1,a_2,\ldots,a_m\), find \(\sum_{i=0}^{m} a_i\).

Solution Note that \(a_0\) represents the number of 0’s in the sequence, \(a_1\) represents the number of 1’s in the sequence, and so on. This gives us that \(a_0 + a_1 + \cdots + a_{n-1} + a_m\) represents the number of values in the sequence between 0 and \(n\). Clearly, \(0 \leq a_i \leq m + 1\) for all \(0 \leq i \leq m\) as the number of \(i\)’s in the sequence is nonnegative and cannot be more than the number of values in the sequence, which is \(m - 0 + 1 = m + 1\). We now show that \(a_i \neq m + 1\) for all \(0 \leq i \leq m\). If \(a_i = m + 1\), then this would mean that all \(m + 1\) members of the sequence equal \(i\). Then all members of the sequence must be equal to \(i\), but that means \(a_i = i \neq m + 1\). A Contradiction! Thus, all members of the sequence are between 0 and \(m\) so \(\sum_{i=0}^{m} a_i = m + 1\).

Problem 3 Alice and Betty are playing a game. First, they pick two positive integers \(k\) and \(n\). They write down the number \(n\) on a whiteboard. On a players turn, she may replace \(n\) with the number \(n-k^m\) where \(m\) is a nonnegative integer and \(n \geq k^m\). A player loses if she cannot make a move. Let Alice make the first move. For which \(n\) does Betty have the winning strategy?

Solution We first note that if \(k\) is odd, then \(k^m\) must be odd as well. This means that for Betty to win (game ends in an even number of moves), \(n\) must be even as well. If \(k\) is even, then we have some more work. First, we note that for all \(n < k\), we have that Betty wins if \(n\) is even as the only possible move is to replace \(n\) with \(n-1\). Because we know the winning states (states where the person currently making a move has a winning strategy) for \(n < k\) (which are all \(n\) such that \(n \equiv 2i - 1 \pmod{k-1}\) and \(n \equiv 0 \pmod{k-1}\) for some positive integer \(i < \frac{k-1}{2}\)), we can show that Betty wins if \(n \equiv 2i \pmod{k-1}\) for some positive integer \(i < \frac{k-1}{2}\). We prove our result using strong induction by assuming that all numbers up to \(n - 1\) follow this rule. Because of the fact that \(k^m \equiv 1 \pmod{k-1}\), every winning state is only a winning state if \(n - 1 \equiv 2i \pmod{k-1}\) for some positive integer \(i < \frac{k-1}{2}\). As a result, we show for all \(n\) that the only \(n\) for which Betty has the winning strategy for odd \(k\) are \(n \equiv 2i \pmod{k-1}\) for positive integers \(i < \frac{k-1}{2}\). Thus, combining our two results for even and odd \(k\), we get that Betty has the winning strategy if and only if \(n \equiv 2i \pmod{k-1}\) for \(i < \frac{k-1}{2}\) for some positive integer \(i\). If \(k\) is odd, we also have the additional solution \(n \equiv 0 \pmod{k-1}\).
Problem 4 On acute triangle $\triangle ABC$, there is a point $P$ on $BC$. Find points $X$ and $Y$ on $AB$ and $AC$ respectively such that the perimeter of $\triangle PXY$ is minimized.

Solution We reflect $P$ across $AB$ and $AC$ to get the points $S$ and $T$ respectively. Note that $AP = AS$, $AX = AX$, and $\angle BAP = \angle BAS$. This means that $\triangle ABP \cong \triangle ABS$ so $XP = SX$. Similarly, we have that $YP = TY$. This means that the perimeter of $\triangle PXY$ is equal to $XP + XY + YP = SX + XY + YT$. Also, for a fixed point $P$, we have that $S$ and $T$ are fixed so the minimum possible perimeter is the minimum length of $SX + XY + YT = ST$. Thus, if we take $X$ and $Y$ to be the intersections of $ST$ with $AB$ and $AC$, we are finished.

Problem 5 For reals $a$, $b$, $c$, $d$, $e$, we have that $a + b + c + d + e = 1$. Show that

$$a^4 + b^4 + c^4 + d^4 + e^4 \geq \frac{1}{125}$$

and find when equality occurs.

Solution 1 We start out with the Cauchy-Schwartz Inequality:

$$(a^2 + b^2 + c^2 + d^2 + e^2)(1 + 1 + 1 + 1 + 1) \geq (a + b + c + d + e)^2 = 1$$

$$\Rightarrow a^2 + b^2 + c^2 + d^2 + e^2 \geq \frac{1}{5}$$

We then do this again:

$$(a^4 + b^4 + c^4 + d^4 + e^4)(1 + 1 + 1 + 1 + 1) \geq (a^2 + b^2 + c^2 + d^2 + e^2)^2 = \frac{1}{25}$$

$$\Rightarrow a^4 + b^4 + c^4 + d^4 + e^4 \geq \frac{1}{125}$$

Solution 2 We note that $f(x) = x^4$ is a convex function so we can use Jensen’s inequality.

$$a^4 + b^4 + c^4 + e^4 \geq 5f\left(\frac{a + b + c + d + e}{5}\right) = 5f\left(\frac{1}{5}\right) = \frac{1}{125}$$