Count the Same Thing in Different Ways!

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Key ideas
What does a binomial coefficient, \( n \) choose \( k \), really mean?
How can we count?
• directly
• one-to-one correspondence
• tree diagram
• multiplication and division for overcounting
• inclusion-exclusion

Problems (based on Ch 5 of Proofs That Really Count by Benjamin and Quinn)
1. The Basics?
   a. What is a binomial coefficient?
   b. Why is it called that, and what other names does it have? (choose, path-counting, combinations, others?)
   c. What is a formula for computing these things?
   d. Where does that formula come from? (One way is to put \( n \) things in order, picking one at a time or picking a group of \( k \) first and then sorting.)
   e. What about “multichoose” where your subsets can have repetition?

2. Multiple Explanations!
   a. Why is \( \binom{n}{k} = \binom{n}{n-k} \)?
   b. Why is \( \binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1} \)?
   c. Why is the total, for all possible \( k \), of \( \binom{n}{k} = 2^n \)?
   d. Why is the total, for all possible even \( k \), of \( \binom{n}{k} = 2^{n-1} \)?
   e. What does that mean about odd \( k \)? And thus about the alternating sum? Does this lead to more possible explanations?
   f. Why is \( k \binom{n}{k} = n \binom{n-1}{k-1} \)?
g. Why is the total, for all possible $k$, of $k \binom{n}{k} = n \cdot 2^{n-1}$?

h. What do we get if we divide both sides of the previous statement by $2^n$?

i. Why is the total, for all possible $j$, of $\binom{m}{j} \binom{n}{k-j} = \binom{m+n}{k}$?

j. If $x$ and $y$ are integers, why is the total for all possible $k$ of $\binom{n}{k} x^k y^{n-k} = (x+y)^n$?

k. Why does $\binom{n}{k} \binom{k}{m} = \binom{n}{m} \binom{n-m}{k-m}$?

l. Why is the total, for all $m < n$, of $\binom{m}{k} = \binom{n+1}{k+1}$? (Hockey stick)

m. Why is the total, for all possible $m$, of $\binom{m}{k} \binom{n-m}{k} = \binom{n+1}{2k+1}$?

3. Multichoosing
   a. How does multichoosing relate to choosing?
   b. Which relationships from the above have analogies to multichoose?

4. Homework?
   a. Why is the total, for all possible $k$, of $k \left( \binom{n}{k} \right)^2 = n \binom{2n-1}{n-1}$?

   b. What is the sum of the squares of the binomial coefficients?
   c. Generalize (g) above: What if you have $k(k-1)$ times the binomial coefficients? Extend further by multiplying in $(k-2)$ and so on.

   d. Why is the total, for all possible $k$, of $\binom{n}{k} \binom{k}{m} = \binom{n}{m} \cdot 2^{n-m}$?

   e. Why is the total, for all possible $k$, of $\binom{n}{2k} \binom{2k}{m} = \binom{n}{m} \cdot 2^{n-m-1}$?

5. Extra challenges?
   a. How many paths from $(0,0)$ to $(n,n)$ never go above the main diagonal?
   b. How many ways are there to write an (unordered) sum of at most $a$ positive integers each of which is at most $b$? For instance, with $a = 2$, $b = 3$, we have $3+3$, $3+2$, $3+1$, $3$, $2+2$, $2+1$, $2$, $1+1$, and the empty set summing to 0, for a total of 10 ways.