Instructions: Work as many problems as you can. Even if you can’t solve a problem, try to learn as much as you can about it. Please write a complete solution to each problem you solve, as if you were entering it into a math contest and had no ability to explain it to the grader in person. This will help you make sure that you’ve thought of all the possibilities.

1. Suppose we have three identical bricks. How can we find the length of the diagonal of a brick (i.e. the distance between two vertices not belonging to the same face) using only one measurement with a ruler?

2. Find all pairs of natural number \((a, b)\) such that \(a^{1000} + 1\) is divisible by \(b^{619}\), and \(b^{1000} + 1\) is divisible by \(a^{619}\). The natural numbers are positive integers and do NOT include zero. Prove that you have found ALL the pairs.

3. How many integers between 2 and 100 inclusive cannot be written as \(m \cdot n\), where \(m\) and \(n\) have no common factors and neither \(m\) nor \(n\) is equal to 1? Note that there are 25 primes less than 100.

4. A number has been obtained by rearranging the digits of another number. (a) Can the sum of these two numbers equal 9,999? (b) Can it equal 99,999?

5. How many subsets \(A\) of \(\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}\) have the property that no two elements of \(A\) sum up to 11?

6. Let \(ABC\) be an equilateral triangle with height 13, and let \(O\) be its center. Point \(X\) is chosen at random from all points inside \(ABC\). Given that the circle of radius 1 centered at \(X\) lies entirely inside \(ABC\), what is the probability that this circle contains \(O\)?

7. A group of \(n\) people are standing in a circle. We number them consecutively clockwise from 1 to \(n\). Starting with person \#2, we remove every other person, proceeding clockwise. For example, if \(n = 6\), the people are removed in the order 2, 4, 6, 3, 1, and the last person remaining is \#5. Let \(j(n)\) denote the last person remaining.
   (a) Compute \(j(n)\) for \(n = 2, 3, \ldots 25\).
   (b) Find a way to compute \(j(n)\) for any positive integer \(n > 1\). You may not get a “nice” formula, but try to find a convenient method that is easy to compute by hand or machine.