Russian Math Circle Class Problems

December 5, 2012

Instructions: Work as many problems as you can. Even if you can’t solve a problem, try to learn as much as you can about it.

1. Is it possible to measure exactly 4 liters of water using a faucet, one 3-liter beaker and one 5-liter beaker?

2. How can you cut 27 inches from a 144-inch ribbon without a ruler?

3. Can the square of an integer end with 2?

4. Find the remainder when: (a) $3^{100}$ is divided by 5; (b) $5^{100}$ is divided by 3.

5. Can the sum of four consecutive natural numbers ever be divisible by 4?

6. Is it true that if $N \geq 6$, a square can always be cut into $N$ smaller squares? The smaller squares do not have to be all the same size.

7. Find the smallest 10-digit number that contains all of the digits from 0 to 9 and is divisible by 36. (A 10-digit number cannot begin with a “0”.)

8. Suppose that $a$, $b$ and $c$ are natural numbers such that $a + b + c$ is divisible by 30. Show that $a^5 + b^5 + c^5$ is also divisible by 30.

9. Prove that a $10 \times 10$ chessboard cannot be completely covered with 25 copies of the polyomino on the left. Can it be covered by 25 copies of the polyomino on the right?

10. Suppose that 100 points are chosen on the segment $AB$ such that the points are symmetric about the midpoint of $AB$. Exactly half of the points are colored red and the others are colored blue. Show that the sum of the distances from $A$ to the red points is the same as the sum of the distances from $B$ to the blue points.

11. Olga and Ludmilla play the following game. Suppose that there are 1001 coins in a pile. A move consists of removing $p^k$ coins from the pile where $p$ is a prime number and $k$ is a non-negative integer. The first person to empty the pile wins. If Olga must move first and both play optimally, which player has a winning strategy? Why?

12. A petri dish contains 2012 bacteria and at midnight, a single virus is added to the dish. At the end of each minute, every virus particle kills a single bacterium. Then the number of virus particles and the number of bacteria double. Will the virus eventually eliminate all the bacteria? If not, prove it. If so, when?
Russian Math Circle Class Solutions

1. Is it possible to measure exactly 4 liters of water using a faucet, one 3-liter beaker and one 5-liter beaker?

**Solution:** Three solutions. Let A be 3L, B be 5L.

(a) Fill A, move A to B, fill A, move 2L from A to B, empty B. Move 1L from A to B. Fill A, 1 + 3 = 4.

(b) Fill B, move 3L from B to A. Empty B. Fill B. Move 1L from B to A. 5 − 1 = 4.

(c) If the beakers are cylindrical, each can be filled to exactly half by tilting the beaker until the surface of the water exactly touches the rim of the beaker and the opposite side of the bottom. Then you have 1.5L in one and 2.5L in the other. Combine them to make 4L.

2. How can you cut 27 inches from a 144-inch ribbon without a ruler?

**Solution:** Fold the long piece in half 4 times, leaving a length of 9. Cut off 3 adjacent 9-inch pieces.

3. Can the square of an integer end with 2?

**Solution:** No. An integer must end with a final digit of 0 through 9. The squares of those end in 0, 1, 4, 9, 6, 5, 6, 9, 4, 1, respectively. There are no 2's.

4. Find the remainder when: (a) $3^{100}$ is divided by 5; (b) $5^{100}$ is divided by 3.

**Solution:** For the first part, look at successive powers of 3, modulo 5. Starting with $3^0$ we have:

\[1, 3, 4, 2, 1, 3, 4, 2, 1, \ldots\]

The pattern cycles every 4, with $3^0, 3^4, 3^8, 3^{12}, \ldots$ leaving remainder of 1. Since 100 is a multiple of 4, the remainder is 1.

Similarly for powers of five:

\[1, 2, 1, 2, 1, 2, 1, 2, \ldots\]

By similar reasoning, the remainder when $5^{100}$ is divided by 3 is also 1.

Both of the results above are based on the following theorem: To calculate remainders, it is convenient to use a general theorem which is easiest to introduce with examples. The remainder on dividing 78 by 5 is 3, and the remainder on dividing 79 by 5 is 4. The remainder on dividing 157 = 78 + 79 by 5 is the same as the remainder on dividing 3 + 4 = 7 by 5, which is 2. The remainder on dividing 6162 = 78 · 79 by 5 is the remainder on dividing 3 · 4 = 12 by 5, which is 2. It does not matter what number you are dividing by to get the remainders, it does not matter what numbers are being divided, and it works with addition, multiplication, and, when correctly understood, subtraction. When subtracting, one can get negative numbers, but the remainder is still supposed to be nonnegative. The remainder when $-8$ is divided by 5 is 2 because $-8 = (-2) \cdot 5 + 2$. When we speak of the remainder upon division by $n$ we require it to be non-negative: one of 0, 1, 2, …, $(n − 1)$.

5. Can the sum of four consecutive natural numbers ever be divisible by 4?

**Solution:**

(a) Consider simplest case, 1, 2, 3, 4. Sum is 10. Ten not divisible by 4. Consider 2, 3, 4, 5, then each number is increased by 1, so the sum increases by 4, so it will have the same remainder as before. We can repeat this argument to show that the sum of four consecutive numbers can never be divisible by 4.

(b) Consider $N, N + 1, N + 2, N + 3$. Sum is $4N + 6$. $4N$ is divisible by 4, but 6 is not.
6. Is it true that if \(N \geq 6\), a square can always be cut into \(N\) smaller squares? The smaller squares do not have to be all the same size.

**Solution:** If we can find a way to do this for \(n = 6, 7, 8\) we are done, since we can always add 3 more squares by cutting an existing square into 4 parts in the obvious way (adding 3 to the total number of squares). The figure below shows how to obtain 6, 7 and 8 squares:

![Figure showing 6, 7, and 8 squares](image)

7. Find the smallest 10-digit number that contains all of the digits from 0 to 9 and is divisible by 36. (A 10-digit number cannot begin with a “0”.)

**Solution:** We know that the ten digits add to 45, so every such 10-digit number will be a multiple of 9. To be a multiple of 36 all we have to do is make sure it is also divisible by 4. We want to use the largest possible final pair of digits to do this and a little fiddling shows that 96 does the trick. So the best answer is:

1023457896.

8. Suppose that \(a\), \(b\) and \(c\) are natural numbers such that \(a + b + c\) is divisible by 30. Show that \(a^5 + b^5 + c^5\) is also divisible by 30.

**Solution:** We just need to show that \(a^5 + b^5 + c^5\) is divisible by 2, 3 and 5 and we’re in business.

Since \(a + b + c\) is divisible by 30 it is also divisible by 2, 3 and 5. It is easy to verify that for any number \(n\), \(n^5\) is equal to \(n\), mod 2, 3 or 5. So if \(a + b + c\) is zero mod 2, 3 or 5 so will be \(a^5 + b^5 + c^5\).

9. Prove that a 10 \(\times\) 10 chessboard cannot be completely covered with 25 copies of the polyomino on the left. Can it be covered by 25 copies of the polyomino on the right?

![Polyominoes](image)

**Solution:** For the polyomino on the left, if the chessboard is colored with alternating black and white squares, there will be 50 of each color. If the polyominoes are similarly colored, each will have either three blacks and a white or three whites and a black.

There are 25 total, so if \(x\) of them have three blacks, then \((25 - x)\) will have one black, for a total count of black squares: \(3x + (25 - x) = 2x + 25\), which is odd, and cannot be equal to 50.

For the polyomino on the right, number the squares on the chessboard as follows:
Every polyomino must cover each of the four numbers 1, 2, 3, 4 exactly once, but the numbering does not have 25 of each number (in particular, there are 26 copies of 2 and 24 copies of 3), so it is impossible.

10. Suppose that 100 points are chosen on the segment $AB$ such that the points are symmetric about the midpoint of $AB$. Exactly half of the points are colored red and the others are colored blue. Show that the sum of the distances from $A$ to the red points is the same as the sum of the distances from $B$ to the blue points.

**Solution:** If a symmetric pair of points have opposite colors, then their contribution of distances to $A$ and $B$ are the same. The number of pairs with the same colors have to include an equal number of red and blue pairs, and each pair adds the same total distance to one side or the other, ensuring a balance.

To see that any symmetric same-color pair adds the same total distance to either point, imagine that the midpoint of $AB$ is at the origin with $A$ at $-1$ and $B$ at 1 and compute the distance of two arbitrary (but symmetrically placed) points to either $A$ or $B$.

11. Olga and Ludmilla play the following game. Begin with 1001 coins in a pile. A move consists of removing $p^k$ coins from the pile where $p$ is a prime number and $k$ is a non-negative integer. The first person to empty the pile wins. If Olga moves first and with optimal play, which player has a winning strategy?

**Solution:** One way to solve this problem is to make a list, starting at 1 coin, whether the person to move can win from that position. Since $1 = 3^0$, $2 = 2^1$, $3 = 3^1$, $4 = 2^2$ and $5 = 5^1$, all piles with 1 through 5 coins can be won immediately by the person whose move it is.

If there are 6 coins, this pile cannot be reduced to 0 in one move, but must be moved to one of the 5 “unsafe” positions, so if you begin with a pile of 6, you lose. That means for piles with 7 through 11 coins, the first person can remove 1 through 5, leaving the other person with a losing position.

Continuing in this way, all piles with a multiple of 6 coins are unsafe for the person to move. Since 1001 is not a multiple of 6 Olga wins. (Of course the strategy above may take a long time. Olga has much faster ways to win, but at least the strategy above will always work.)

12. A petri dish contains 2012 bacteria and at midnight, a single virus is added to the dish. At the end of each minute, every virus particle kills a single bacterium. Then the number of virus particles and the number of bacteria double. Will the virus eventually eliminate all the bacteria? If not, prove it. If so, when?

**Solution:** If we just make a list of the number of bacteria at the beginning of each minute beginning at midnight, it is easy to see what is going on:
<table>
<thead>
<tr>
<th>Time</th>
<th>Virus</th>
<th>Bacteria</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$2^0$</td>
<td>$(2012) \cdot 2^0$</td>
</tr>
<tr>
<td>1</td>
<td>$2^1$</td>
<td>$(2011) \cdot 2^1$</td>
</tr>
<tr>
<td>2</td>
<td>$2^2$</td>
<td>$(2010) \cdot 2^2$</td>
</tr>
<tr>
<td>3</td>
<td>$2^3$</td>
<td>$(2009) \cdot 2^3$</td>
</tr>
<tr>
<td>4</td>
<td>$2^4$</td>
<td>$(2008) \cdot 2^4$</td>
</tr>
</tbody>
</table>

It is clear that after 2012 minutes, there will be zero bacteria.