Plane Geometry and Recent BAMO Brilliance Solutions

San Jose Math Circle

by Zvezdelina Stankova
Berkeley Math Circle Director

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1. Warm-Ups

Problem 1. (Three Squares) Three identical squares with bases $AM$, $MH$, and $HB$ are put next to each other to form a rectangle $ABCD$. Find the sum of the angles $\angle AMD + \angle AHD + \angle ABD$ and prove that your answer is correct.

Problem 2. (Research Problem) Generalize Problem #1 to 4 squares. Can you find the sum of the resulting four angles? How about the same problem for 5 squares? How about $n$ squares? What happens when we let $n$ go to infinity (i.e., we use an infinite number of squares): will the sum of the angles be a finite angle, or will all angles add up to infinity?

Problem 3. (Farmer and Cow) During a hot summer day, a farmer and a cow find themselves on the same side of a river. The farmer is 2 km from the river and the cow is 6 km from the river. If each of them would walk straight to the river, they would find themselves 4 km from each other. Unfortunately, the cow has broken its leg and cannot walk. The farmer needs to get to the river, dip his bucket there, and take the water to the cow. To which point on the river should the farmer walk so that his total walk to the river and then to the cow is as short as possible?

Problem 4. (Shortest Broken Line) Two lines $p_1$ and $p_2$ intersect. Two points $A$ and $B$ lie in the acute angle formed by the lines. Find a point $C$ on $p_1$ and a point $D$ on $p_2$ so that the broken line $ADBCA$ has the smallest possible length. Prove that the points you have found indeed yield this smallest possible length.

Problem 5. (Minimal Perimeter) Given $\triangle ABC$, on the ray opposite to ray $\overline{CA}$ take a point $B_1$ so that $|CB_1| = |CB|$. Prove that

(a) Point $B_1$ is the reflection of $B$ across the angle bisector $l$ of the exterior angle of the triangle at vertex $C$.

(b) If $D$ is an arbitrary point on $l$ different from $C$, the perimeter of $\triangle ABD$ is bigger than the perimeter of $\triangle ABC$.

Problem 6. (Find the Perimeter) In $\triangle ABC$, $|AC| = |BC|$ and $|AB| = 10$ cm. Through the midpoint $D$ of $AC$ we draw a line perpendicular to $AC$. This line intersects $BC$ in point $E$. The perimeter of $\triangle ABC$ is 40 cm. Find the perimeter of $\triangle ABE$.

Problem 7. (Locating Angle Bisector) Two points $A$ and $B$ and a line $l$ are given so that the line intersects segment $AB$ (neither $A$ nor $B$ lies on $l$). Find point $C$ on $l$ so that the angle bisector of $\angle ACB$ lies on $l$. Prove that your construction is correct.

Problem 8. (The Bridge) Two villages $A$ and $B$ lie on opposite sides of a straight river of width $d$ km. There will be a market in village $B$, which residents of village $A$ wish to attend. To this end, the people of village $A$ need to build a bridge across the river so that the total route walked by the residents of $A$ to the bridge, across the bridge, and onward to $B$ is as short as possible. The bridge, of course, has to be built perpendicular to the river. How can the villagers from $A$ find the exact location of the bridge?
2. Preparation in Circle Geometry

Problem 9. Distances to Inscribed Equilateral Triangle) An equilateral triangle is inscribed in a circle. An arbitrary point is selected on the circle. The distances from this point to two of the vertices of the triangle are 4 cm and 7 cm, respectively. What could the distance from the point to the third vertex be? Give all possible answers and prove your claim.

Problem 10. (Distances in an Inscribed Quadrilateral) A quadrilateral is inscribed in a circle. The two diagonals of the quadrilateral are 9 cm and 6 cm long. Three of the sides of the quadrilateral, going clockwise around the circle, are 8 cm, 3 cm, and 3 cm long. How long is the fourth side of the quadrilateral? Give all possible answers and prove your claim.

Problem 11. (Medians and Angle Bisectors) Let $CM$ be a median in $\triangle ABC$, and $CP$ – a median in $\triangle AMC$. Given that $AB = 2AC = 2CM$, prove that $CM$ is the angle bisector of $\angle PCB$.

Problem 12. (Altitudes and Circumcircle) $\triangle ABC$ is inscribed in circle $k$. The extensions of the two altitudes $AE$ and $BD$ of $\triangle ABC$ intersect $k$ in points $A_1$ and $B_1$, respectively. If $\angle C = 60^\circ$, prove that $AA_1 = BB_1$. Is the converse true?

Problem 13. (Napoleon’s Theorem) Externally to $\triangle ABC$ are drawn three equilateral triangles $ABP$, $BCQ$, and $CAN$, as well as the circles described about each of these equilateral triangles.

(a) Prove that the three circles intersect in a point.
(b) Prove that the lines $AQ$, $BN$, and $CP$ also intersect in a point.
(c)* Prove that the centers of the three circles form an equilateral triangle.

Problem 14. (BAMO ’07) In $\triangle ABC$, $D$ and $E$ are two points inside side $BC$ such that $BD = CE$ and $\angle BAD = \angle CAE$. Prove that $\triangle ABC$ is isosceles.

Problem 15. (BAMO ’10) Acute $\triangle ABC$ has $\angle BAC < 45^\circ$. Point $D$ lies in the interior of $\triangle ABC$ so that $BD = CD$ and $\angle BDC = 4\angle BAC$. Point $E$ is the reflection of $C$ across line $AB$, and point $F$ is the reflection of $B$ across line $AC$. Prove that lines $AD$ and $EF$ are perpendicular.

Problem 16. (BAMO ’11) Three circles $k_1$, $k_2$, and $k_3$ intersect in point $O$. Let $A$, $B$, and $C$ be the second intersection points (other than $O$) of $k_2$ and $k_3$, $k_1$ and $k_3$, and $k_1$ and $k_2$, respectively. Assume that $O$ lies inside of the triangle $ABC$. Let lines $AO$, $BO$, and $CO$ intersect circle $k_1$, $k_2$, and $k_3$ for a second time at points $A'$, $B'$, and $C'$, respectively. If $|XY|$ denotes the length of segment $XY$, prove that

$$\frac{|AO|}{|AA'|} + \frac{|BO|}{|BB'|} + \frac{|CO|}{|CC'|} = 1.$$