Part I: Remainders and Divisibility

What remainders can you get if you divide a square by four? Why?

Show that the number 100,000,003 is not the sum of two squares.

Show that if a number has an odd number of divisors, then it is a perfect square.

Show that the number 111,111,111,111 is not a square.

Show that the polynomial \( n^3 - 16n \) is always divisible by three.

Find the last digit of two to the power of three to the power of four to the power of five, \( 2^{3^{4^5}} \).

Without resorting to a counting argument, show that the product of any five consecutive numbers must be divisible by 120.

Show that \( n^2 + 2n + 18 \) can never be a square.

Show that \( x^3 + y^3 + 4 \) cannot be a cube.

Challenge – show that \( n^4 + 8n^2 + 21 \) can never be a square.

Part 2: Pigeonhole Principle

I have three kinds of socks in a drawer. How many socks do I need to pull out to make sure that I have a pair?

I have twenty five crates containing three types of bananas. I wish to make \( N \) gift baskets, each containing the same type of bananas. What is the largest \( N \) can be such I will be sure to be able to make the baskets?

Sixteen people choose a random integer. Prove that two must differ by a multiple of fifteen.

Show that in a group of ten people, there must be two people who have an identical number of friends.

Show that in a group of fifty-two numbers, there are two whose squares differ by a multiple of a hundred.

Part 3: Other Problems (some related, some not)

Show that you cannot cover a equilateral triangle with two smaller equilateral triangles.

A hundred numbers are sitting in a circle. Each number differs from its neighbors by no more than one. Prove that there are two numbers sitting across from each other which differ by one or less.

How many queens can you put on a chessboard such that none attack each other? Kings? Show.

Three tangent marbles of radius one are on a table. A marble of radius two sits on them. How high off the ground is the center of the large marble? What if a marble of radius one sits on a marbles of radius two? A marble of radius one on marbles of radius three? Four?