Mathematical Induction

Principle of Mathematical Induction (English)

1. Show something works the first time.
2. Assume that it works for this time,
3. Show it will work for the next time.
4. Conclusion, it works all the time

Principle of Mathematical Induction (Mathematics)

1. Show true for \( n = n_0 \) (base of induction)
2. Assume true for \( n = k \), where \( k > n_0 \)
3. Show true for \( n = k + 1 \)
4. Conclusion: Statement is true for all \( n \geq n_0 \)

The key word in step 2 is assume. You are not trying to prove it's true for \( n = k \), you're going to accept on faith that it is, and show it's true for the next number, \( n = k + 1 \). If it later turns out that you get a contradiction, then the assumption was wrong.

**PROBLEMS**

1. Use mathematical induction to prove that the following statements are true for all positive integers \( n \) and \( m \):

   a. \( 1 + 2 + 3 + \ldots + n = n(n + 1)/2 \)

   b. \( 1^2 + 2^2 + 3^2 + \ldots + n^2 = n(n + 1)(2n + 1)/6 \)

   c. \( 1^3 + 2^3 + 3^3 + \ldots + n^3 = n^2(n + 1)^2/4 \)

   d. \( 2^{n+3} + 5^n \times 3^{n+2} \) is divisible by 17

   e. \( n^{2m-1} + 1 \) is divisible by \( n + 1 \)

   f. \( n^3 + 2n \) is divisible by 3

   g. \( \frac{1}{1 \times 3} + \frac{1}{3 \times 5} + \frac{1}{5 \times 7} + \ldots + \frac{1}{(2n-1)(2n+1)} = \frac{n}{2n+1} \)

2. Prove that:

   a. \( n! \geq 2^n \) for \( n = 4, 5, 6 \ldots \)

   b. \( n! \geq 3^n \) for \( n = 7, 8, 9 \ldots \)

   c. \( 3^n \geq n^2 \) for \( n = 1, 2, 3, \ldots \)

   d. \( 2^n \geq n \) for \( n = 1, 2, 3 \ldots \)

   e. \( 2^n \geq n^2 \) for \( n = 4, 5, 6 \ldots \)
3. Given:

\[ x + \frac{1}{x} \text{ - is an integer} \]

Prove:

\[ x^n + \frac{1}{x^n} \text{ - is also an integer} \]

4. Bernoulli’s inequality

Prove that for all \( a > -1 \):

\[ (1 + a)^n \geq 1 + na \]

5. Prove that any side of a
a. quadrilateral
b. pentagon
c. any polygon
is less than the sum of all other sides

6. Prove that in order to cut an \( n \)-sided polygon in a set of triangles by drawing non-intersecting diagonals you have to draw exactly \( n-3 \) diagonals.

7. Find mistakes in the following proofs.

A. \( n > n + 1 \)

Proof. Let’s assume that it is true for \( n = k \), i.e. \( k > k + 1 \). Let’s prove that it is true for \( n = k + 1 \). By adding 1 to both sides we get:

\[ k + 1 > (k + 1) + 1 = k + 2 \]

B. All cows in any herd are the same color.

Proof. If there is one cow in a herd the statement is true. Assume that it is true for a herd with \( k \) cows and let’s prove it for a herd with \( k + 1 \) cows. Pick a cow A, then the rest of the cows have the same color (by assumption). Out of these cows, let’s pick a cow B. So the rest of the cows including A have the same color and this will be the same color as B. So all cows have the same color.

C. In a country each city is connected via a highway to at least one other city.

Prove that it is possible to travel between any two cities by a car.

Proof. If there is only one city, there is nothing to prove. Let’s assume that the statement is true for any country with no more than \( k \) cities and let’s prove it for a country with \( k + 1 \) cities. Let’s pick a city A. There is a way to travel between any two other cities in a country based on assumption. When we add city A back, it has to be connected to at least one other one. So it will be connected to all other cities in the country as well.
8. Prove that there is a way to tile the squares of the sizes 4x4, 8x8, 16x16, etc. with one cell missing in any position using the L-shaped tiles – squares 2x2 with one missing cell.

9. Mocha island is divided into countries. It turns out that every country is triangular, and all countries which share a border share that entire side of the triangle. Prove that there is a way to color a map of Mocha island using three colors such that no two neighboring countries are the same color.

10. Let's consider $n$ lines in a plane. They divide the plane into a number of regions. Prove that you there is a way to paint all of the regions using 2 colors in such a way that no neighbors share the same color.

11. On Mocha island there are freeways and railways. Every two cities on this island are connected by a railway or freeway or both. Prove that there is a way to get from any city to any other city on this island by either using a car or train, i.e. you can travel everywhere by using only one means of transportation.

12. There are $n$ lines in a plane. None of them are parallel and no three of them intersect in a same point. How many regions do they divide the plane into?

13. The vertices of a convex polygon are painted in exactly three colors in such a way that no neighbors are painted in the same color. Prove that it is possible to draw the diagonals in such a way they split the polygon into triangles such that all vertices of each triangle are painted different colors.

14. There are two kinds of students in a class – lefties and righties. Each lefty has at least one friend who is a righty. When a lefty has an even number of righty friends in the classroom she will talk incessantly; if she has an odd number of righty friends she will be silent. Righties never talk at all. Show that the teacher can kick out less than half of her class in such a way that the rest of the class is silent.