Russian Math Circle Problems

April 1, 2011

Instructions: Work as many problems as you can. Even if you can’t solve a problem, try to learn as much as you can about it. Please write a complete solution to each problem you solve, as if you were entering it into a math contest and had no ability to explain it to the grader. This will help you make sure that you've thought of all the possibilities.

1. Suppose \( S \) is a set and \( * \) is a binary operation on \( S \). (This means that for any two elements \( x \) and \( y \) in \( S \), the element \( x * y \) is also in \( S \).) The only things we know about \( * \) are these (in particular, \( * \) is not necessarily associative):
   
   \[ x * x = x \quad \text{for all } x \in S \]
   \[ (x * y) * z = (y * z) * x \quad \text{for all } x, y, z \in S \]
   
   Show that \( x * y = y * x \) for all \( x, y \in S \).

2. The number \( 26! = 26 \cdot 25 \cdot 24 \cdots 3 \cdot 2 \cdot 1 \) ends with a long string of zeroes. Let \( N \) be the number that remains after all those zeros are removed from the end. Find the largest value of \( k \) such that \( 12^k \) divides \( N \).

3. Suppose two circles with equal radius \( r \) are inscribed as illustrated in the right triangle below having side lengths of 3, 4 and 5. Find \( r \).

![Diagram of right triangle with inscribed circles]

4. Find the number of ways to choose 1005 numbers from the set \( \{1, 2, 3, \ldots, 2009, 2010\} \) in such a way that the sum of any two chosen numbers is neither 2010 nor 2011.

5. Coin turning problems. In every problem below there are \( n \) coins placed on a table with “heads” showing. Each move requires you to turn over \( k \) different coins. The goal is to finish with all showing “tails”. For each situation below, show how to accomplish the goal or prove that it is impossible.
   
   (a) \( n = 7 \) and \( k = 2 \).
   (b) \( n = 5 \) and \( k = 3 \).
   (c) \( n = 6 \) and \( k = 4 \).
   (d) What can you say about general \( n \) and \( k \)?
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1. Compute:
   \[ \sqrt{(111, 111, 111, 111)(1, 000, 000, 000, 005)} + 1. \]

2. Let \( n \) be a positive integer.
   (a) If \( 2n + 1 \) is a perfect square, show that \( n + 1 \) is the sum of two perfect squares.
   (b) If \( 3n + 1 \) is a perfect square, show that \( n + 1 \) is the sum of three perfect squares.

3. In the following figure, let the circle \( k \) centered at \( O \) be perpendicular to the line \( l \) at \( B \) with \( AOB \) being a diameter of \( k \). Choose a point \( M \) on \( k \) and extend \( AM \) to intersect \( l \) at \( C \). Construct a line tangent to \( k \) at \( M \) and let that line intersect \( l \) at \( D \). Show that \( |BD| = |DC| \).

4. Find a set of positive integers \( a_1, a_2, a_3, \ldots, a_n \) such that \( a_1 + a_2 + a_3 + \cdots + a_n = 1000 \) and such that the product \( a_1 \cdot a_2 \cdot a_3 \cdots a_n \) is as large as possible.

5. Suppose we begin with a \( 7 \times 7 \) square grid of points as in the figure below.

(a) How many squares can be formed whose vertices are grid points? Remember to include squares like the one in the figure.

(b) What if we begin with an \( n \times n \) grid?
1. Suppose $S$ is a set and $*$ is a binary operation on $S$. (This means that for any two elements $x$ and $y$ in $S$, the element $x * y$ is also in $S$.) The only things we know about $*$ are these (in particular, $*$ is not necessarily associative):

\[ x * x = x \quad \text{for all } x \in S \]

\[ (x * y) * z = (y * z) * x \quad \text{for all } x, y, z \in S \]

Show that $x * y = y * x$ for all $x, y \in S$.

**Solution:** From the first rule, we have:

\[ x * y = (x * y) * (x * y). \]

We will now apply the rules above repeatedly to convert the term on the right gradually to $(y * x)$:

\[
(x * y) = (x * y) * (x * y) \\
= (y * (x * y)) * x \\
= ((x * y) * x) * y \\
= (((y * x) * x) * y) * y \\
= ((x * x) * y) * y \\
= (x * y) * y \\
= (y * y) * x \\
= y * x.
\]

2. The number $26! = 26 \cdot 25 \cdot 24 \cdot \cdots \cdot 3 \cdot 2 \cdot 1$ ends with a long string of zeroes. Let $N$ be the number that remains after all those zeros are removed from the end. Find the largest value of $k$ such that $12^k$ divides $N$.

**Solution:** In $26!$ there are six factors of 5, twenty three factors of 2 and ten factors of 3. There will be six zeros at the end of $26!$ and removing them will leave seventeen factors of 2 and still ten factors of 3. Each factor of 12 in the remaining number requires two factors of 2 and one of 3, so we are limited by the seventeen factors of 2. Thus the number will be divisible by $12^8$ and not $12^9$, so $k = 8$.

3. Suppose two circles with equal radius $r$ are inscribed as illustrated in the right triangle below having side lengths of 3, 4 and 5. Find $r$. 

![Diagram of a right triangle with two inscribed circles]
Solution: Drop perpendiculars to the sides of the triangle, connect the centers of the inscribed circles, construct lines through the centers of the circles that meet to form a triangle and label the points as shown below:

Let $|CD| = h_1$, $|DA| = h_2$, $|BG| = l_1$, $|GA| = l_2$. Obviously, $|CD| = |CE|$ and $|BG| = |BF|$. We also can see that $|HJ| = h_2 - r$, that $|IJ| = l_2 - r$, that $|HI| = 2r$ and that $\triangle H1J \sim \triangle CBA$.

From the diagram and the similarity of the large and small triangle, we have the following equations:

1. $5 = l_1 + h_1 + 2r$
2. $3 = h_1 + h_2$
3. $4 = l_1 + l_2$
4. $\frac{4}{3} = \frac{l_2 - r}{h_2 - r}$
5. $\frac{5}{3} = \frac{2r}{h_2 - r}$

We now have five equations and five unknowns, and with a bit of relatively easy algebra, we can calculate the value of $r$.

From Equation 5 we obtain $h_2 = \left(\frac{11}{5}\right)r$.

From Equation 3 we find $l_2 = 4 - l_1$ and we can substitute for $l_2$ and $h_2$ in Equation 4 to obtain:

$$4\left(\frac{11}{5}r - r\right) = 3(4 - l_1 - r)$$

$$\frac{39}{5}r = 12 - 3l_1$$

$$l_1 = 4 - \frac{13}{5} r. \quad (6)$$

From Equation 2 and the value of $h_2$ in terms of $r$, we have:

$$h_1 = 3 - h_2 = 3 - \frac{11}{5} r. \quad (7)$$

Substituting the values of $h_1$ and $l_1$ from Equations 6 and 7 into Equation 1, we obtain:

$$5 = 4 - \frac{13}{5} r + 3 - \frac{11}{5} r + 2r$$

$$-2 = -\frac{14}{5} r$$

$$\frac{5}{7} = r.$$
4. Find the number of ways to choose 1005 numbers from the set \{1, 2, 3, \ldots, 2009, 2010\} in such a way that the sum of any two chosen numbers is neither 2010 nor 2011.

**Solution 1:**
Let’s form 1005 pairs: \{1, 2010\}, \{2, 2009\}, \{3, 2008\}, \ldots, \{1005, 1006\}. Clearly, a set of chosen numbers must contain exactly one number from each of these pairs. Let \(S\) be a chosen set, and suppose that \(x \leq 2009\) is an element of \(S\). Then \(2010 - x\) is not chosen, and hence the second number in the pair \{2010 - x, y\} must be chosen. But it’s obvious that \(y = x + 1\). Therefore, \(x + 1\) is also in \(S\). Let \(n\) be the smallest number in \(S\). If \(n \leq 1005\), repeating the above argument we can see that \(S\) must contain \(n, n + 1, \ldots, 1005\), and the larger of the two elements of each of the pairs \{1, 2010\}, \{2, 2009\}, \ldots, \{n - 1, 2012 - n\}, so that:

\[
S = \{n, n + 1, n + 2, \ldots, 1005, 2010, 2009, \ldots, 2012 - n\}.
\]

Thus we have exactly 1005 such sets for \(n = 1, 2, \ldots, 1005\). The only other possible solution is:

\[
S = \{1006, 1007, \ldots, 2010\}.
\]

Hence the answer is 1006.

**Solution 2:**
We’ll illustrate the solution with a smaller example, but one with the same characteristics. Suppose the set just contained the numbers \{1, 2, 3, \ldots, 8\} and we want to find subsets of 4 of them such that no pair in the subset adds to either 8 or 9. In the diagram below all the numbers are listed, and two numbers are connected by a line segment if the two cannot appear in the same set. For example, 6 is connected to both 2 and 3 since \(6 + 2\) and \(6 + 3\) yield 8 or 9.

![Diagram](image)

The numbers 8 and 4 are special in that they are the only numbers that eliminate only one other possibility. There are 7 line segments in the drawing and if we do not choose either 4 or 8, our 4 chosen numbers each will have 2 lines coming out which will require 8 total lines and there are only 7 available. Thus we are required to choose either 4 or 8 in our set.

If we choose, say, 4 (choosing 8 is symmetric), then 5 cannot be in the set, and we might as well erase the line from 3 to 5, so we have a zig-zag pattern that is identical, but two numbers shorter, from which we need to choose 3 numbers. The same argument applies: we need to choose one from one of the two remaining ends, et cetera.

We can see that any final set must contain \(k\) (possibly zero) numbers starting from the 8 and continuing right along the lower line and \((4 - k)\) numbers beginning with 4 and continuing left along the upper line. For this situation, there are solutions for \(k = 0, 1, 2, 3, 4\), so there will be exactly 5 subsets:

\[
\{8, 7, 6, 5\}, \{8, 7, 6, 4\}, \{8, 7, 3, 4\}, \{8, 2, 3, 4\}, \{1, 2, 3, 4\}.
\]

The same argument can be used for the original problem, and there are 1006 subsets satisfying the conditions.
5. Coin turning problems. In every problem below there are \( n \) coins placed on a table with “heads” showing. Each move requires you to turn over \( k \) different coins. The goal is to finish with all showing “tails”. For each situation below, show how to accomplish the goal or prove that it is impossible.

(a) \( n = 7 \) and \( k = 2 \).
(b) \( n = 5 \) and \( k = 3 \).
(c) \( n = 6 \) and \( k = 4 \).

(d) What can you say about general \( n \) and \( k \)?

**Solution:** This is an open-ended problem, so this solution is not complete, but here are some ideas:

(a) This is impossible, since turning two coins will leave the number of heads odd. The goal is zero heads which is not odd. This is an example of an invariant: “The number of heads remains odd no matter which two coins are turned.”

(b) The can be done. Turn 1, 2, 3, then 3, 4, 5, then 1, 2, 4. The net result leaves all the coins heads except we have turned over coin 5. Thus we have a scheme that will turn over exactly one coin. By renumbering the coins, we can repeat the process five times and turn all of them. This is clearly not the most efficient way to do it, but it does provide a proof that the result is possible.

(c) Turn 1, 2, 3, 4, then 2, 3, 4, 5, and finally 2, 3, 4, 6.

(d) This is the open-ended part. Here are some strategies to consider:

(a) Parity, as used in part (a).

(b) Given any sequence of moves, the net result will be that some of the coins are turned and some will remain the same. If you can find a sequence that leaves all the coins except one the same, you can eventually turn all of them, as used in part (b).

(c) The previous rule shows that if you can find a way to turn exactly one coin for some pair \((n, k)\), then you can solve the situation for any \((m, k)\), where \( m \geq n \). For example, with \( n = 4 \) and \( k = 3 \), the sequence 1, 2, 3, then 2, 3, 4, then 1, 2, 4 turns only coin 2, so by choosing different sets of 4 coins to apply this method, every set of coins with \( n \geq 3 \) can be turned by some sequence of 3-at-a-time turns.

Here is a complete solution, thanks to Julian Ziegler Hunt:

We proceed under the natural assumptions that \( k, n \in \mathbb{Z}^+ \) and \( n \geq k \), as otherwise the problem would make little sense. If \( k = n \), then it is clearly possible to complete our objective in a single move. Now, suppose that \( n \neq k \), so that \( k + 1 \leq n \). Suppose that we label two of our coins \( a \) and \( b \). Then pick \( k - 1 \) other coins (which we can do since \( k + 1 \leq n \)), and label them \( x_1, \ldots, x_{k-1} \). Then perform two moves: first flip the coins labeled \( a, x_1, \ldots, x_{k-1} \), then flip the coins labeled \( b, x_1, \ldots, x_{k-1} \). The net result of performing the two moves is that \( a \) and \( b \) each got flipped, and all of the other coins remained the same, so we can, in two moves, flip an arbitrary pair of coins. This suggests that we must take the parity of \( n \) and \( k \) into account.

In our first case, we suppose that \( n \) is even. Then we can split the \( n \) coins into \( \frac{n}{2} \) pairs of coins, then flip each pair of coins individually by the process described above.

In the second case, \( n \) is odd and \( k \) is even. Here, we cannot achieve our objective (all \( n \) coins flipped to tails), since each time we flip a coin, the number of tails changes from even to odd or odd to even, so that flipping \( k \) coins (an even number) preserves the parity of the number of tails, so that there is always an even number of tails. But \( n \) is odd, so our goal is to end up with an odd number of tails, which is impossible.
Our third case is where \( n \) and \( k \) are both odd. Here, we can first flip \( k \) coins, ending up with an even number of coins left to flip, then splitting these into pairs and flipping each pair individually.

This solves the original problem, as stated.

We can actually figure out the minimum number of moves required to achieve our goal with only a little difficulty, so we shall do so.

First, a lemma similar to the one involving flipping any given pair of coins: if, after two moves, exactly \( q \) coins have been flipped, then \( q = 2r \) and \( k + r \leq n \) for some \( r \leq k \), and any \( 2r \) coins with \( r \leq k \) and \( k + r \leq n \) can be flipped in exactly two moves. For the first part, suppose that the coins flipped by each move had exactly \( s \) coins in common. Then, letting \( r = k - s \), we have that those \( s \) coins remained the same, the other \( r \) flipped by each move changed, and the coins not affected by either move remain the same. Thus there were exactly \( 2r \) coins flipped, there were \( s + 2r = k + r \) coins that were flipped by one of the moves so that \( k + r \leq n \), and \( s \geq 0 \) so \( r \leq k \).

For the second part, suppose we are given \( 2r \) coins which are to be flipped, with \( r \leq k \) and \( k + r \leq n \). Then split the \( 2r \) coins into two sets of \( r \) coins, and pick some other \( k - r \) coins (which can be done since \( r \leq k \) and \( k + r \leq n \)). Then, for our first move, we flip the first set of \( r \) coins and the \( k - r \) coins, and for our second, we flip the second set of \( r \) coins and the \( k - r \) coins. These are both valid moves, and they have the net result of flipping the given \( 2r \) coins. Thus the lemma is proved.

Now, to compute the minimum number of moves, we unfortunately must break into a number of cases (fortunately, most of them are fairly simple).

**Case 1:** If \( n \) is odd and \( k \) is even, and in this case we proved above that we can’t achieve our goal, so the minimum number of moves is \( \infty \).

**Case 2:** If \( n = k \), here the minimum number of moves is 1.

**Case 3:** If \( k < n < 2k \), we will split this case into two sub-cases. We clearly can’t do this in one move, and if we could do it two moves, then we would have \( n = 2r, r \leq k, k + r \leq n \), but then \( n = 2r \leq k + r \leq n \), so \( r = k \) and \( n = 2k \), a contradiction.

**Case 3a:** If \( n \) and \( k \) have the same parity, we can do it in three moves, since we can first flip some \( k \) coins, be left with \( n - k \) coins to flip, which is an even number with \( \frac{n - k}{2} \leq k \) and \( \frac{n - k}{2} \leq n \), so we can finish in another two moves.

**Case 3b:** If, instead, \( k \) is odd and \( n \) is even, then the same parity argument used to show that \( n \) odd, \( k \) even shows that it can’t be done in an odd number (such as three) of moves (the parity of the number of tails changes every move, so after an odd number of moves it would be odd, but \( n \) is even). Thus, we can consider a solution as a sequence of Moves, each of which consists of two normal moves. Each Move flips any number of the form \( 2r \) with \( r \leq n - k \), so the smallest number of Moves is achieved by doing as many Moves that flip \( 2(n - k) \) coins as possible, then perhaps one more Move to flip the remaining coins. This number is given by \( \left\lceil \frac{n}{2(n-k)} \right\rceil \), where \( \lceil x \rceil \) is the unique integer such that \( \lceil x \rceil - 1 < x \leq \lceil x \rceil \), so the minimum number of moves is \( 2 \left\lceil \frac{n}{2(n-k)} \right\rceil \).

**Case 4:** If \( n \geq 2k \), and in this situation we must consider three sub-cases, depending on the parities of \( n \) and \( k \). First, though, in any case, the minimum number of moves is at least \( \left\lceil \frac{n}{2k} \right\rceil \), since each move can flip at most \( k \) coins from heads to tails.

**Case 4a:** If \( n \) and \( k \) are both even, then we flip \( k \) heads to tails each move until there are at most \( 2k \) heads left, at which point, by the lemma (since the number of heads remaining is even and \( n \geq 2k \geq k + r \)), we can finish in two more moves, for a total of \( \left\lceil \frac{n-2k}{2k} \right\rceil + 2 = \left\lceil \frac{n}{2k} \right\rceil \) moves.
Case 4b: If $n$ is even and $k$ is odd, then the number of moves must be even (see above), so we split any solution into two-move Moves, and try to solve for the minimum number of Moves. A Move can flip any even number $\leq 2k$ of coins (since $k + r \leq 2k \leq n$ for all $r \leq k$), so we minimize the number of Moves by doing $\left\lceil \frac{n}{2k} \right\rceil - 1$ Moves that flip $2k$ coins, and one more Move that flips somewhere between 2 and $2k$ coins, for a total of $\left\lceil \frac{n}{2k} \right\rceil$ Moves, or $2 \left\lceil \frac{n}{2k} \right\rceil$ moves.

Case 4c: If $n$ and $k$ are both odd, we do a single move, so that we need only flip $n - k$ more coins, and then, by identical logic to the previous case, this takes an additional $2 \left\lceil \frac{n-k}{2k} \right\rceil$ moves, for a total of $2 \left\lceil \frac{n-k}{2k} \right\rceil + 1$ moves. The results can be summed up in the following table:

<table>
<thead>
<tr>
<th></th>
<th>even</th>
<th>odd</th>
</tr>
</thead>
<tbody>
<tr>
<td>even</td>
<td>1 if $n = k$</td>
<td>$\infty$</td>
</tr>
<tr>
<td></td>
<td>3 if $k &lt; n &lt; 2k$</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$\left\lceil \frac{n}{k} \right\rceil$ if $2k \leq n$</td>
<td></td>
</tr>
<tr>
<td>odd</td>
<td>1 if $n = k$</td>
<td>1 if $n = k$</td>
</tr>
<tr>
<td></td>
<td>4 if $k &lt; n &lt; 2k$</td>
<td>3 if $k &lt; n &lt; 2k$</td>
</tr>
<tr>
<td></td>
<td>$2 \left\lceil \frac{n-k}{2k} \right\rceil$ if $2k \leq n$</td>
<td>$2 \left\lceil \frac{n-k}{2k} \right\rceil + 1$ if $2k \leq n$</td>
</tr>
</tbody>
</table>
1. Compute:

\[ \sqrt{(111,111,111,111)(1,000,000,000,005) + 1}. \]

**Solution:** Let \( n = 1,000,000,000,000 \). Then we can re-write the expression above as follows:

\[
\sqrt{\frac{(n - 1)(n + 5)}{9}} + 1 = \frac{1}{3} \sqrt{(n - 1)(n + 5) + 9} \\
= \frac{1}{3} \sqrt{n^2 + 4n + 4} \\
= \frac{1}{3} \sqrt{(n + 2)^2} \\
= \frac{1}{3} (n + 2) = 333,333,333,334.
\]

2. Let \( n \) be a positive integer.

(a) If \( 2n + 1 \) is a perfect square, show that \( n + 1 \) is the sum of two perfect squares.

(b) If \( 3n + 1 \) is a perfect square, show that \( n + 1 \) is the sum of three perfect squares.

**Solution:**

(a) Since \( 2n + 1 \) is odd, it must be the square of an odd number, say \( (2k + 1)^2 = 2n + 1 \). Then \( 4k^2 + 4k + 1 = 2n + 1 \), or \( 2k^2 + 2k = n \). Thus \( n + 1 = (k^2) + (k^2 + 2k + 1) \), or \( n + 1 = (k^2) + (k + 1)^2 \): the sum of two squares.

(b) Since \( 3n + 1 = x^2 \) is a perfect square, then \( x \equiv \pm 1 \pmod{3} \), so \( x = 3m \pm 1 \). Thus \( 3n + 1 = (3m \pm 1)^2 = 9m^2 \pm 6m + 1 \). We have \( n = 3m^2 \pm 2m \), so

\[ n + 1 = 3m^2 \pm 2m + 1 = m^2 + m^2 + (m^2 \pm 2m + 1) = m^2 + m^2 + (m \pm 1)^2. \]

3. In the following figure, let the circle \( k \) centered at \( O \) be perpendicular to the line \( l \) at \( B \) with \( AOB \) being a diameter of \( k \). Choose a point \( M \) on \( k \) and extend \( AM \) to intersect \( l \) at \( C \). Construct a line tangent to \( k \) at \( M \) and let that line intersect \( l \) at \( D \). Show that \( |BD| = |DC| \).

**Solution:** Construct the line \( BM \). Since the lines \( DB \) and \( DM \) are tangent to the circle from \( D \), we have \( |DM| = |BD| \). This makes \( \triangle BDM \) isosceles, so \( \angle DBM = \angle DMB \). Since \( \angle AMB \) is inscribed in a semicircle, it is a right angle, so \( \triangle BMC \) is a right triangle. Thus \( \angle DBM + \angle MCD = 90^\circ \) and
\( \angle DMB + \angle DMC = 90^\circ \), so a little algebra gives us \( \angle MCD = \angle DMC \). So \( \triangle CDM \) is also isosceles, and \( |DM| = |DC| \).

Since \( |DM| = |DC| \) and \( |DM| = |BD| \) we have \( |BD| = |DC| \).

4. Find a set of positive integers \( a_1, a_2, a_3, \ldots, a_n \) such that \( a_1 + a_2 + a_3 + \cdots + a_n = 1000 \) and such that the product \( a_1 \cdot a_2 \cdot a_3 \cdots a_n \) is as large as possible.

**Solution:** Since we are looking for a large product, it is pointless to have any of the \( a_i = 1 \). We can reduce \( n \) by 1 and add that value of 1 to any of the other numbers to leave the sum the same and increase the value of the product, so we know that for all \( i \), we have \( a_i \geq 2 \).

Similarly, if any of the \( a_i \geq 5 \) we can obtain a larger product by breaking it as close to in half as possible, thus in the maximum product, for all \( i \) we have \( 2 \leq a_i \leq 4 \).

Any solution containing a 4 is equivalent to a solution with two 2’s, so any optimal solution can be made to contain only 2’s and 3’s. If there are three 2’s in a product, we can do better if we replace them with two 3’s, since \( 2 \times 2 \times 2 < 3 \times 3 \). Thus we seek a solution with as many 3’s as possible. We could have up to 333 of them, but that leaves an extra 1, so the best solution is either 332 copies of 3 and either one 4 or two 2’s.

5. Suppose we begin with a \( 7 \times 7 \) square grid of points as in the figure below.

(a) How many squares can be formed whose vertices are grid points? Remember to include squares like the one in the figure.

(b) What if we begin with an \( n \times n \) grid?

**Solution:**

For every \( k \times k \) square block of vertices, there are exactly \( k - 1 \) squares (tilted and not) that will fit exactly in that square and in no smaller square. See the figure below.

In an \( n \times n \) grid, we need to find out how many \( k \times k \) square grids fit inside and are aligned with the edges of the larger grid. These are determined by the position of the upper-left corner of the smaller squares, and it’s easy to see that there are \( (n - k + 1)^2 \) upper-left corners. To fit any squares inside, \( k \) has to be at least two.

Thus, the total number of squares that can be drawn, tilted or not, is given by:

\[
1 \cdot (n - 1)^2 + 2 \cdot (n - 2)^2 + 3 \cdot (n - 3)^2 + \cdots + (n - 1) \cdot 1^2.
\]
This formula can be proven to be equal to \( n^2(n^2 - 1)/12 \).

For \( n = 7 \), this yields:

\[
1 \cdot 6^2 + 2 \cdot 5^2 + 3 \cdot 4^2 + 4 \cdot 3^2 + 5 \cdot 2^2 + 6 \cdot 1^2 = 49 \cdot 48/12 = 196.
\]