

Russian-style Session Homework Problems

Instructions: Work as many problems as you can. Even if you can't solve a problem, try to learn as much as you can about it. Please write a complete solution to each problem you solve, as if you were entering it into a math contest and had no ability to explain it to the grader in person. This will help you make sure that you've thought of all the possibilities.

1. Let ABC be a triangle with $\angle A > 90^\circ$. On side BC , two distinct points P and Q are chosen so that $\angle BAP \cong \angle PAQ$ and $BP \cdot CQ = BC \cdot PQ$. Find $\angle PAC$.
2. To print all the page numbers of a book required 1890 individual digits. How many pages are in the book?
3. Let $S(x)$ denote the sum of the digits of x (in decimal representation). Do there exist three different natural numbers m, n , and q such that $m + S(m) = n + S(n) = q + S(q)$?
4. Each vertex of a cube is assigned one of the numbers 1 or -1 . Each face of the cube is assigned the number equal to the product of the numbers at the four vertices on that face. Is it possible that the sum of the 14 assigned integers is 0?
5. Find all triples (a, b, c) of positive integers such that $\left(1 + \frac{1}{a}\right)\left(1 + \frac{1}{b}\right)\left(1 + \frac{1}{c}\right) = 3$.
6. Suppose that a, b, c , and d are positive integers that satisfy the following equations:

$$\begin{cases} ab + cd = 38 \\ ac + bd = 34 \\ ad + bc = 43 \end{cases}$$

Find the value of $a + b + c + d$.

7. As shown in the figure below, an equilateral triangle with side t , a square of side s , and a circle of radius r touch each other in a right triangle ABC with side $BC = 1$. Find t .

