THE GEOMETRY OF MEDIANS

1. Let ABC be a triangle with $BC = a$, $AC = b$, and $AB = c$. Let m_a, m_b, and m_c be the lengths of the medians from A, B, and C of triangle ABC. Prove that:

   (i) $\dfrac{b+c-a}{2} < m_a < \dfrac{b+c}{2}$.

   (ii) Prove that in any triangle the sum of the medians is greater than $\dfrac{3}{4}$ of the perimeter but less than the perimeter.

   (iii) Prove that $m_a^2 = \dfrac{2(b^2+c^2)-a^2}{4}$ and conclude that $m_a^2 + m_b^2 + m_c^2 = \dfrac{3(a^2+b^2+c^2)}{4}$.

   (iv) Prove that $m_a = \dfrac{a}{2}$ if and only if triangle ABC is right with $\angle A = 90^\circ$.

   (v) Prove that $a = b$ if and only if $m_a = m_b$.

   (vi) Prove that if $a > b$, then $m_a < m_b$.

   (vii) Prove that $a^2 + b^2 > \dfrac{1}{2}c^2$, and deduce that $m_a^2 + m_b^2 > \dfrac{9}{8}c^2$.

   (viii) Let $x = ab + bc + ca$, and let $x_1 = m_a m_b + m_b m_c + m_c m_a$. Prove that: $\dfrac{9}{20} < \dfrac{x_1}{x} < \dfrac{5}{4}$.

   (ix) Let AA_1 and BB_1 be the medians from A and B. Prove that $AA_1 \perp BB_1$ if and only if $a^2 + b^2 = 5c^2$.

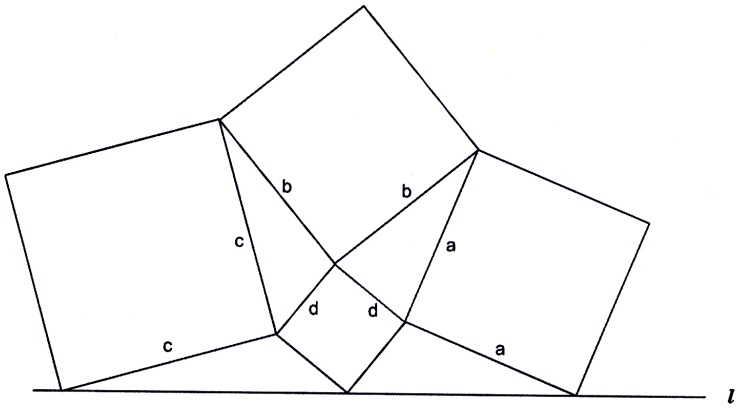
2. *(1975 AHSME, ♯20)* Triangle ABC is such that $AB = 4$ and $AC = 8$. If M is the midpoint of BC and $AM = 3$, what is the length of BC?

   (A) $2\sqrt{26}$ (B) $2\sqrt{31}$ (C) 9 (D) $4 + 2\sqrt{13}$

   (E) Not enough information given to solve the problem

3. *(2008 AIME II, ♯5)* In trapezoid $ABCD$ with $\overline{BC} \parallel \overline{AD}$, let $BC = 1000$ and $AD = 2008$. Let $\angle A = 37^\circ$ and $\angle D = 53^\circ$, and let M and N be the midpoints of $\overline{BC}$ and $\overline{AD}$, respectively. Find the length of MN.

4. Three squares of sides a, c, and d touch the line ℓ, and they each have one vertex in common with a square of side b, as shown below. If $d = 4$, find b.

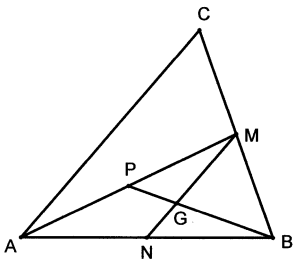


5. Let G be the centroid of triangle ABC. Prove that if triangles ABG, BCG, and ACG have the same perimeter, than $\triangle ABC$ is equilateral.

6. *(1976 AHSME, ♯9)* In triangle ABC, D is the midpoint of AB, E is the midpoint of DB, and F is the midpoint of BC. If the area of triangle ABC is 96, then the area of triangle AEF is:

   (A) 16 (B) 24 (C) 32 (D) 36 (E) 48

7. *(1996 Mandelbrot, Round 3 Individual, ♯4)* Triangle ABC has area 48. Let P be the midpoint of median $\overline{AM}$ and let N be the midpoint of side $\overline{AB}$. If G is the intersection of $\overline{MN}$ and $\overline{BP}$, find the area of triangle MGP.



8. Let G be the centroid of triangle ABC. Prove that triangles ABG, BCG, and CAG have the same area.

9. *(1957 AHSME, ♯26)* From a point P within a triangle, line segments are drawn to the vertices. A necessary and sufficient condition that the three triangles formed have equal areas is that the point P be:

   (A) the center of the inscribed circle (B) the center of the circumscribed circle

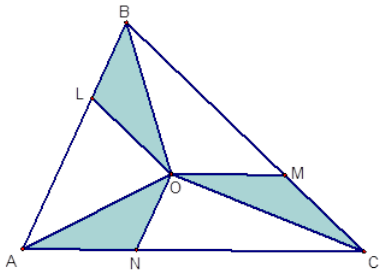
   (C) such that the three angles formed at P each be 120° (D) the intersection of

   the altitudes of the triangle (E) the intersection of the medians of the triangle.

10. Given triangle ABC, find all points P such that the areas of triangles ABP, BCP, and ACP are equal.

11. Prove that the medians divide any triangle into six triangles of equal area.

12. Inside given triangle ABC find a point O such that the areas of triangles BOL, COM, and AON are equal (points L, M, and N lie on sides AB, BC, and CA such that $OL \parallel BC$, $OM \parallel AC$, and $ON \parallel AB$).



13. *(2000 MOSP)* Prove that the area of a triangle whose sides are equal to the medians of a triangle of area S is equal to $\frac{3}{4}S$.

14. *(1991 Japan Mathematical Olympiad)* Given that G is the centroid of triangle ABC , $GA = 2\sqrt{3}$, $GB = 2\sqrt{2}$ and $GC = 2$, find the area of triangle ABC.

15. *(1962 AHSME, ♯39)* The medians AN and BP of a triangle with unequal sides are, respectively, 3 inches and 6 inches long. Its area is $3\sqrt{15}$ square inches. The length of the third median, in inches, is:

   (A) 4 (B) $3\sqrt{3}$ (C) $3\sqrt{6}$ (D) $6\sqrt{3}$ (E) $6\sqrt{6}$

16. *(2004 USAMTS, Round 1, ♯5)* Point G is where the medians of $\triangle ABC$ intersect, and point D is the midpoint of the side $\overline{BC}$. The triangle BDG is equilateral with side length 1. Determine the lengths AB, BC, and CA of the sides of triangle ABC.

17. *(The Yearly Competition of Gazeta Matematica, 1991, 9th and 10th grade, Romania)* Let M, N, and P be the midpoints of sides $\overline{AB}$, $\overline{BC}$, and $\overline{CA}$ of the triangle ABC, respectively. On the perpendicular bisectors of $\overline{AB}$, $\overline{BC}$, and $\overline{CA}$ choose, inside the triangle, points A', B', and C' such that $\dfrac{MC'}{AB} = \dfrac{NA'}{BC} = \dfrac{PB'}{AC}$. Show that the triangles ABC and $A'B'C'$ have the same centroid.

18. *(1986 AIME, ♯15)* Let $\triangle ABC$ be a right triangle in the xy-plane with the right angle at C. Given that the length of the hypotenuse AB is 60, and that the medians through A and B lie along the lines $y = x+3$ and $y = 2x+4$, respectively, find the area of $\triangle ABC$.