

Introduction to Counting II

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First we saw counting by casework. We could count the ways to win on a 3x3 tic-tac-toe board by counting three vertical, three horizontal, and two diagonal. Extending that, we found $10 + 10 + 4 + 4$ ways to win with four in a row on a 5x5 board, by counting the ways to win in each of four directions, $-$, $|$, \backslash , and $/$.

When the board size got to be 19x19, we saw that there were some nice shortcuts to count: count based on the starting point of the horizontal, vertical, or diagonal piece. That way, we saw that there were 19x15 places where a horizontal five-in-a-row could begin, and similarly 15x19 for a vertical. There are 15x15 for a \backslash , and 15x15 for a $/$.

For the 4x4x4, finding 4-in-a-row got trickier. We did it by several different methods, some of which seemed at first to give different answers until we realized what kinds of overcounting were taking place. One method is to count the number of lines passing through each spot in the grid: the eight corners and the eight center spaces have 7 lines, and the remaining 48 spots have 4 lines, for a total of $16 \times 7 + 48 \times 4 = 304$ lines. But then each line has been counted four times, once for each spot along it, so there are actually $304/4 = 76$ lines.

We then counted the lines in a different way: there are 16 top-to-bottom straight lines, 16 left-to-right, and 16 front-to-back, which is 48 so far. While counting those, we also saw there are 12 different plane slices through the cube, 4 in each of three directions, and each slice has 2 diagonals, so that's 24 more lines. Finally there are 4 space diagonals (one from each of the top corners, connecting to the opposite corner on the bottom). $48 + 24 + 4 = 76$.

Then I suggested that we count differently: each of those 12 plane slices has 10 lines, 4 horizontal, 4 vertical, and 2 diagonal. That makes 120. Then there are the 4 space diagonals, so that makes 124. What went wrong? We double-counted each of the 48 straight lines, since they are part of two slices. And $124 - 48$ is, thankfully, exactly 76.

We added up all the numbers in a 10x10 multiplication table by using the distributive property: $(1+2+3+4+5+6+7+8+9+10) \times (1+2+3+4+5+6+7+8+9+10)$. This turned out to be related to the question of counting the rectangles formed by the grid lines of that board. If you count the number of rectangles whose bottom right square is the current spot, you find that the multiplication is the same as the number of possibilities for the top left corner of that rectangle. There's also an alternative way of counting the rectangles: count how many are of each size. For instance, there are 9x10 places where you can put a 2x1 rectangle, and 10x9 places where you can put a 1x2 rectangle, and so on. You discover the same sum in a different way.

Finally, we counted the number of ways to write a number as the sum of triangular numbers, using dynamic programming in much the same way as in the previous class. First we had to decide exactly what we are counting: Can we use 0 as a triangular number? Does the order matter? It turns out that the most interesting counting problem, because of some unexpected connections to other counts, is to include 0 and count the different orders as different.

If you can use one triangular number, then there's 1 way to make 0, 1, 3, 6, 10, ..., and 0 ways for any other number. But once we have that, we're almost done! To count the number of ways to make, say, 16 with three triangular numbers, you go through the casework: the first number could be 0, so you count the number of ways to make 16 with two triangular numbers. Or it could be 1, so add the number of ways to make 15 with two triangular numbers. And so on, adding the number of ways for 13, 10, 6, and 1. If you keep track of what you've calculated in an organized list, you get a lot of sums, and that's all you have to do to produce all these calculations.

Tic-Tac-Toe

1. How many ways are there to win in ordinary tic-tac-toe?
2. How many ways are there to win with four in a row on a 5x5 board?
3. How many ways are there to win with five in a row on a 19x19 board?
4. How many ways are there to win with four in a row on a 4x4x4 board?

Multiplication Tables and Rectangles

5. What is the sum of the 100 numbers in a standard 10x10 multiplication table?
6. How many rectangles are formed by the lines in your multiplication table?

Sums and Dynamic Programming

7. How many ways are there to write 10 as the sum of four squares?
What possible different interpretations are there for this problem?
How hard does it get? Could we go all the way up to 100 with our methods?
Just for fun, if you have a list of the number of ways of writing n as the sum of four squares, compare these numbers with the sum of the positive divisors of n . What do you learn?
8. How many ways are there to write 10 as the sum of four triangular numbers?
What possible different interpretations are there for this problem?
How hard does it get? Could we go all the way up to 100 with our methods?
Just for fun, if you have a list of the number of ways of writing n as the sum of four squares, compare these numbers with the sum of the positive divisors of $2n + 1$. What do you learn?

Some Counting Exercises

9. What is the smallest possible number of weekend days in a year? What is the largest?
10. How many isosceles triangles have a perimeter of 42 inches, with sides whose lengths are a whole number of inches?
11. How many isosceles triangles have a perimeter of 420 inches, with sides whose lengths are a whole number of inches?
12. Three kids share 30 chocolate bars at Halloween in such a way that each kid gets a prime number of chocolate bars. Is this possible? How many ways are there to do it?