

Introduction to Counting I

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Joshua Zucker, joshua.zucker@stanfordalumni.org

Key ideas

First of all we had a fundamental counting strategy for identifying how many items there are in a list: transform the list to one that starts 1, 2, 3, ... and then you know the last number also tells you how many numbers there are in the list. The general strategy of transforming things into simpler things is very powerful! Here we might take a list like 9, 11, 13, ..., 45 and first subtract 7 from every number and then divide by 2 to get 1, 2, 3, ..., 19. There are 19 odd numbers in the range 9 through 45.

A second key strategy we used was to break up a complicated collection to count into simpler collections. In problem 1e we can count based on the place value of the leftmost digit 1. Then we can further break that down into the group that starts 10 and the group that starts 11. In problem 2, it helps a lot to break things into, for example, the ways that include no dimes and the ways that include at least one dime.

The third major lesson was that we can often reuse results from previous problems in order to help solve the next one. For instance, the group that starts 10 with the first 1 in the millions place corresponds perfectly to all the list that has its first 1 in the hundred thousands place: just delete that first one and then stick it on after the 0. Similarly, the ways to pay 30 cents with at least one dime is the same as the ways to pay 20 cents using any coins whatsoever: just add a dime to any of those collections.

Finally, perhaps it's obvious, but we often matched things up by showing that there was a correspondence between two groups. The correspondences in the previous paragraph are examples. We could also see, for instance, that there were the same number of ways to pay 14 cents as 10 cents, because any way of paying 10 cents can have 4 pennies added to it, and any way of paying 14 cents must contain at least 4 pennies so it can have 4 pennies deleted from it. It's always important to check that your correspondence works both ways, to be sure that there aren't any extra cases that have been left out!

Some counting exercises

1. Let's arrange the numbers 1 through 9 in a square.
 - a. What is the smallest possible sum for a row?
 - b. What is the largest possible sum?
 - c. How many sums are possible?
 - d. What are the answers for parts a-c if we use the numbers 1, 3, 5, 7, ..., 17 instead?
 - e. What if we use the numbers 1, 10, 100, 1000, ..., 100000000?
 - f. What if we examine products instead of sums?

2. How many ways are there to pay exactly \$1.00 using
 - a. Only quarters?
 - b. Only quarters and/or dimes?
 - c. Only quarters, dimes, and/or nickels?
 - d. Only quarters, dimes, and/or pennies?
 - e. Only quarters, dimes, nickels, and/or pennies?

3. \$200 are to be shared between two people so that each person gets a whole number of dollars, and at least \$25. How many ways are there to do this?

4. \$200 are to be shared between two people so that each person gets a whole number of dollars, and neither person gets an amount divisible by \$3 or \$5. How many ways are there to do this?