San Jose Math Circle May 16, 2009

LOGARITHMS

DEFINITION. Let $a > 0$, $a \neq 1$ and let $x > 0$. By the *logarithm of x with base a* we mean the exponent to which a must be raised to get x. In other words:

$$\log_a x = y \quad \Leftrightarrow \quad a^y = x$$

Notice that we can only compute the logarithm of a positive number and that the base of a logarithm is a positive number different from 1.

EXAMPLES. The following examples show that the logarithm of a positive number can have both positive and negative values.

(a) $\log_2 8 = 3$ because $2^3 = 8$.

(b) $\log_{10} 100 = 2$ because $10^2 = 100$.

(c) $\log_6 6 = 1$ because $6^1 = 6$.

(d) $\log_5 1 = 0$ because $5^0 = 1$.

(e) $\log_3 \frac{1}{3} = -1$ because $3^{-1} = \frac{1}{3}$.

(f) $\log_9 3 = \frac{1}{2}$ because $9^{1/2} = \sqrt{9} = 3$.

(g) $\log_2 \frac{1}{4} = -2$ because $2^{-2} = \frac{1}{2^2} = \frac{1}{4}$.

(h) $\log_{25} \frac{1}{5} = -\frac{1}{2}$ because $25^{-1/2} = \frac{1}{25^{1/2}} = \frac{1}{\sqrt{25}} = \frac{1}{5}$.

COMMON LOGARITHMS

Logarithms with base 10 are called *common logarithms* or *decimal logarithms.* We often drop the subscript of 10 for the base when using common logarithms and we write $\log x$ for $\log_{10} x$. Hence:

$$\log x = y \quad \Leftrightarrow \quad 10^y = x$$

Most calculators have a LOG key which can be used to calculate common logarithms.

For example, $\log 4 \approx 0.602$ and $\log 20 \approx 1.301$.

NATURAL LOGARITHMS

It can be shown using calculus that as n increases without bound, the value of the expression $\left(1+\frac{1}{n}\right)^n$ approaches a certain irrational number, denoted by e. The number e arises in the investigation of many physical phenomena. Its value is

$$e = 2.71828\ldots$$

Hence $2 < e < 3$. One can also show that $e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \ldots$.

Logarithms with base e are called *natural logarithms.* The symbol $\ln x$ is used to abbreviate $\log_e x$. Hence:

$$\ln x = y \quad \Leftrightarrow \quad e^y = x$$

Natural logarithms can be evaluated by using the LN key on a calculator.

For example, $\ln 5 \approx 1.609$, $\ln 46 \approx 3.827$, etc.

PROPERTIES OF LOGARITHMS

Let $a > 0$, $a \neq 1$, x, $y > 0$ and let n be any real number. Then the following hold:

(1) $\log_a a = 1$ and $\log_a 1 = 0$

(2) $a^{\log_a x} = x$ and $\log_a a^x = x$

(3) Product Rule: $\log_a(xy) = \log_a x + \log_a y$

(4) Quotient Rule: $\log_a \left(\frac{x}{y}\right) = \log_a x - \log_a y$

(5) Power Rule: $\log_a x^n = n \log_a x$

For the common logarithms, the above rules become:

(1) $\log 10 = 1$ and $\log 1 = 0$

(2) $10^{\log x} = x$ and $\log 10^x = x$

(3) Product Rule: $\log(xy) = \log x + \log y$

(4) Quotient Rule: $\log \left(\frac{x}{y}\right) = \log x - \log y$

(5) Power Rule: $\log x^n = n \log x$

For the natural logarithms, the rules become:

(1) $\ln e = 1$ and $\ln 1 = 0$

(2) $e^{\ln x} = x$ and $\ln e^x = x$

(3) Product Rule: $\ln(xy) = \ln x + \ln y$

(4) Quotient Rule: $\ln\left(\dfrac{x}{y}\right) = \ln x - \ln y$

(5) Power Rule: $\ln x^n = n \ln x$

WARNINGS. As indicated by the following warnings, there are no laws for expressing $\log_a(x+y)$ and $\log_a(x-y)$ in terms of simpler logarithms:

$$\log_a(x+y) \neq \log_a x + \log_a y$$
$$\log_a(x-y) \neq \log_a x - \log_a y$$

THE EXCHANGE PROPERTY

The change of base property (or the exchange property) shows that every logarithm can be expressed in terms of the common or natural logarithm.

If a and b are positive numbers other than 1 and if $x > 0$, then

$$\log_a x = \frac{\log_b x}{\log_b a}$$

In particular, for $b = 10$ and $b = e$, we obtain:

$$\log_a x = \frac{\log x}{\log a} \quad \text{and} \quad \log_a x = \frac{\ln x}{\ln a}$$

Also, if $x = b$, we get: $\log_a b = \dfrac{\log_b b}{\log_b a}$ and since $\log_b b = 1$, we have:

$$\log_a b = \frac{1}{\log_b a} \quad \Rightarrow \quad \log_a b \cdot \log_b a = 1$$

ADDITIONAL USEFUL PROPERTIES

If $a > 0$, $a \neq 1$ and if x_1 and x_2 are positive real numbers, then the following equivalent conditions hold:

(1) If $x_1 \neq x_2$, then $\log_a x_1 \neq \log_a x_2$.

(2) If $\log_a x_1 = \log_a x_2$, then $x_1 = x_2$.

1. Evaluate the following logarithms without using a calculator. Justify your answer in each case:

   (a) $\log_5 1$

   (b) $\log_3 3$

   (c) $\log_7 49$

   (d) $\log_5 125$

   (e) $\log_4 \frac{1}{16}$

   (f) $\log_2 128$

   (g) $\log \frac{1}{10}$

   (h) $\ln e^4$

   (i) $\log_4 \frac{1}{2}$

   (j) $10^{\log 3}$

   (k) $\log 10^{-6}$

   (l) $e^{\ln 8}$

   (m) $\ln e^{2/3}$

2. Simplify the following expressions without using a calculator:

(a) $e^{2\ln 3}$

(b) $e^{3\ln 2 - 2\ln 5}$

(c) $\ln \frac{e^3\sqrt{e}}{e^{1/3}}$

(d) $\log_2 3 \cdot \log_3 4 \cdot \log_4 8$

(e) $\frac{\log_7 243}{\log_7 3}$

(f) $5^{\log_{\frac{1}{5}} \frac{1}{2}}$

(g) $\log_{\frac{1}{\sqrt{2}}} 2 + \log_{\sqrt{2}} 4$

3. Simplify the following real numbers without using a calculator:

(a) $5^{\sqrt{\log_5 7}} - 7^{\sqrt{\log_7 5}}$

(b) $7^{\log_{49} 5-1}$

(c) $343^{1-2\log_{49} 13}$

(d) $\log_{\sqrt{2}} \dfrac{4}{\sqrt{7}+\sqrt{3}} + \log_{\frac{1}{2}} \dfrac{1}{10+2\sqrt{21}}$

(e) $(\log 5)^3 + (\log 20)^3 + \log 8 \cdot \log \dfrac{1}{4}$

4. Simplify the following expressions without using a calculator:

(a) $\dfrac{\log_2 24}{\log_{96} 2} - \dfrac{\log_2 192}{\log_{12} 2}$

(b) $\dfrac{\log_3 12}{\log_{36} 3} - \dfrac{\log_3 4}{\log_{108} 3}$

(c) $\dfrac{\log_3 135}{\log_{15} 3} - \dfrac{\log_3 5}{\log_{405} 3}$

5. If $\log_a b = m$, find $\log_b \sqrt{b\sqrt[3]{ab}}$.

6. If $\log_{14} 7 = a$ and $\log_{14} 5 = b$, express $\log_{35} 28$ in terms of a and b.

7. *(Timisoara Mathematics Review, No. 723, 1971)* If $\log_{40} 100 = a$, calculate $\log_{16} 25$ in terms of a.

8. Given that $\log_{12} 27 = a$, find $\log_6 16$ in terms of a.

9. Given that $\log_{60} 3 = a$ and $\log_{60} 5 = b$, find $\log_{12} 2$ in terms of a and b.

10. Let $a = \log_{45} 25$. Find $\log_{15} 27$ in terms of a.

11. *(1995 AMC 12, ♯24)* There exist positive integers A, B, and C, with no common factor greater than 1, such that

$$A \log_{200} 5 + B \log_{200} 2 = C$$

What is $A + B + C$?

(A) 6 (B) 7 (C) 8 (D) 9 (E) 10

12. Show that if $a = \log_{12} 18$ and $b = \log_{24} 54$, then $ab + 5(a - b) = 1$.

13. Show that if a and b are positive real numbers satisfying $a^2 + b^2 = 7ab$, then

$$\log \frac{a+b}{3} = \frac{1}{2}(\log a + \log b)$$

14. If a, b, c, x, y, $z \in (0,1) \cup (1,\infty)$, then:

$$\log_a x \cdot \log_b y \cdot \log_c z = \log_b x \cdot \log_c y \cdot \log_a z$$

15. Show that $(\log_2 5)^{-1} + (\log_{12} 5)^{-1} < 2$.

16. Show that $\log_7 8 < \log_6 7$.

17. Which is bigger: $3^{\log_2 5}$ or $2^{\log_3 5}$? Justify your answer.

18. Show that $\dfrac{5}{9} < \dfrac{\log 6}{\log 17} < \dfrac{13}{20}$.

19. Prove the following inequalities:

(a) $\log_2 3 + \log_3 4 + \log_4 5 + \log_5 6 > 5$

(b) $2\sqrt{2} < \log_2 3 + \log_3 4 < 3$

20. Prove that for all a, $b \in (0,1)$ the following inequality holds:

$$\log_a \frac{2ab}{a+b} + \log_b \frac{2ab}{a+b} \geq 2$$

21. Show that for all positive real numbers a, b, and c with $ab, bc, ca \neq 1$, the following inequality holds:

$$\log_{bc} a + \log_{ca} b + \log_{ab} c \geq \frac{3}{2}$$

22. Show that if $a, b, c \in (1, \infty)$ or $a, b, c \in (0, 1)$, then:

$$\frac{\log_b a}{a+b} + \frac{\log_c b}{b+c} + \frac{\log_a c}{c+a} \geq \frac{9}{2(a+b+c)}$$

23. Prove that for all $a, b, c \in (0, 1)$ the following inequality holds:

$$\log_a \frac{3abc}{ab+bc+ca} + \log_b \frac{3abc}{ab+bc+ca} + \log_c \frac{3abc}{ab+bc+ca} \geq 3$$