FUNCTIONS AND SEQUENCES

1. *(2003 AMC 12B, ♯9)* Let f be a linear function for which $f(6) - f(2) = 12$. What is $f(12) - f(2)$?

   (A) 12 (B) 18 (C) 24 (D) 30 (E) 36

2. *(2004 AMC 12B, ♯13)* If $f(x) = ax + b$ and $f^{-1}(x) = bx + a$ with a and b real, what is the value of $a + b$?

   (A) -2 (B) -1 (C) 0 (D) 1 (E) 2

3. *(2007 AMC 12B, ♯9)* A function f has the property that $f(3x-1) = x^2 + x + 1$ for all real numbers x. What is $f(5)$?

(A) 7 (B) 13 (C) 31 (D) 111 (E) 211

4. *(2000 AMC 10, ♯24 and 2000 AMC 12, ♯15)* Let f be a function for which $f\left(\frac{x}{3}\right) = x^2 + x + 1$. Find the sum of all values of z for which $f(3z) = 7$.

(A) $-\frac{1}{3}$ (B) $-\frac{1}{9}$ (C) 0 (D) $\frac{5}{9}$ (E) $\frac{5}{3}$

5. *(2001 AMC 12, ♯9)* Let f be a function satisfying $f(xy) = f(x)/y$ for all positive real numbers x and y. If $f(500) = 3$, what is the value of $f(600)$?

(A) 1 (B) 2 (C) $\frac{5}{2}$ (D) 3 (E) $\frac{18}{5}$

6. (*2006 AMC 12A, ♯18*) The function f has the property that for each real number x in its domain, $\frac{1}{x}$ is also in its domain and

$$f(x) + f\left(\frac{1}{x}\right) = x$$

What is the largest set of real numbers that can be in the domain of f?

(A) $\{x \mid x \neq 0\}$ (B) $\{x \mid x < 0\}$ (C) $\{x \mid x > 0\}$

(D) $\{x \mid x \neq -1 \text{ and } x \neq 0 \text{ and } x \neq 1\}$ (E) $\{-1, 1\}$

7. (*Adapted from Gazeta Matematica, Romania*) Let $f(x) = x^2+x+2$ and $g(x) = x^2-x+2$ for all real numbers x. Find all functions h for which

$$(h \circ f)(x) + (h \circ g)(x) = (g \circ f)(x)$$

for all real numbers x.

8. *(2002 AMC 10B, ♯23)* Let $\{a_k\}$ be a sequence of integers such that $a_1 = 1$ and $a_{m+n} = a_m + a_n + mn$ for all positive integers m and n. Then a_{12} is

(A) 45 (B) 56 (C) 67 (D) 78 (E) 89

9. *(1992 AMC 12, ♯18)* The increasing sequence of positive integers a_1, a_2, a_3, $\ldots$ has the property that

$$a_{n+2} = a_n + a_{n+1} \quad \text{for all } n \geq 1$$

If $a_7 = 120$, then a_8 is:

(A) 128 (B) 168 (C) 193 (D) 194 (E) 210

10. (*2004 AMC 10A, ♯24 and 2004 AMC 12A, ♯17*) Let $a_1, a_2, \ldots$ be a sequence with the following properties:

(i) $a_1 = 1$, and

(ii) $a_{2n} = n \cdot a_n$ for any positive integer n.

What is the value of $a_{2^{100}}$?

(A) 1 (B) 2^{99} (C) 2^{100} (D) 2^{4950} (E) 2^{9999}

11. (*2006 AMC 10B, ♯18*) Let $a_1, a_2, \ldots$ be a sequence for which

$$a_1 = 2, \quad a_2 = 3, \quad \text{and} \quad a_n = \frac{a_{n-1}}{a_{n-2}} \quad \text{for each positive integer } n \geq 3$$

What is a_{2006}?

(A) $\frac{1}{2}$ (B) $\frac{2}{3}$ (C) $\frac{3}{2}$ (D) 2 (E) 3

12. (*2004 AMC 10B, ♯19*) In the sequence 2001, 2002, 2003, ... each term after the third term is found by subtracting the previous term from the sum of the two terms that precede that term. For example, the 4th term is $2001 + 2002 - 2003 = 2000$. What is the 2004th term in this sequence:

(A) -2004 (B) -2 (C) 0 (D) 4003 (E) 6007

13. (*1996 BMO, 1^{st} Round*) A function f is defined for all positive integers and satisfies $f(1) = 1996$ and

$$f(1) + f(2) + \cdots + f(n) = n^2 f(n) \quad \text{for all } n > 1$$

Calculate the exact value of $f(1996)$.

14. *(1983 Romanian Olympiad, Final Round)* Determine the function $f : \mathcal{N}^\star \to \mathcal{R}_+^\star$ satisfying the following properties:

(i) $f(4) = 4$

(ii) $\dfrac{1}{f(1)f(2)} + \dfrac{1}{f(2)f(3)} + \cdots + \dfrac{1}{f(n)f(n+1)} = \dfrac{f(n)}{f(n+1)}$ for all $n \in \mathcal{N}^\star$.